A General Schema for Bilateral Proof Rules

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0 Introduction

Logical bilateralism of the sort proposed by Smiley (1996) and Rumfitt (2000) provides rules for both *affirming* and *denying* sentences, prefixing each sentence with a positive or negative force-marker to indicate its affirmation or denial.[1] Formalisms of this sort have been used to provide various proof-theoretically virtuous natural deduction systems for classical logic and, as such, have been prominent in the development of proof-theoretic semantics for classical logic. There are, however, two worries regarding such systems. First, such systems seem to proliferate rules, generally requiring twice as many rules as their unilateral counterparts. Second, such systems seem to provide too much freedom, with a number of different sets of rules being put forward as definitive of the meanings of the classical connectives. In this paper, I respond to both of these worries by proposing a new kind of bilateral sequent calculus in which the rules for all of the classical connectives are given by a single rule schema which, I argue, is uniquely definitive of the meanings of the classical connectives. In this system, each connective is given exactly two rules—a positive rule saying when one is to affirm a sentence with that connective and a negative rule saying when one is to deny a sentence

[1] It's worth noting from the outset that this sort of bilateralism contrasts with the sort of bilateralism proposed by Restall (2005) and Ripley (2013) in which multiple conclusion systems are *interpreted* bilaterally.

with that connective. Bilateral harmony is established by a generalized bilateral analogue to Cut Elimination for any rules of this form. Familiar consequences of Cut Elimination—such as the consistency of the logic and conservativity of the rules—follow. Beyond the intrinsic interest for bilateralism in proof-theoretic semantics, the result is of general interest as it illustrates a new method for formulating systems and doing one's metatheory at a higher level of generality, simplifying both the presentation of systems and the proofs of their key properties.[2]

The paper is structured as follows. In Section One, I lay out the basic bilateralist proposal in proof-theoretic semantics and present the two main worries regarding the number and choice of rules. In Section Two, I introduce a new kind of bilateral proof system—a bilateral sequent calculus—in which all of the connective rules are yielded by a single rule schema. In Section Three I use this system to respond to the two main worries for bilateralism, showing, first, that the worry about the number of rules required in a bilateral system is straightforwardly dealt with and arguing, second, that, in the context of providing a proof-theoretic semantics for the classical connectives, the rules yielded by the schema proposed are to be favored over those yielded by the only plausible alternative schema. In Section Four, I explain why the rules for all of the binary connectives can be specified by a single schema. In the Appendix, I provide proofs of the technical results stated in the body of the paper.

1 The Promise of Bilateralism and Two Worries

Michael Dummett (1991) famously argues that, if we want to think of the meanings of the logical connectives in terms of the rules governing their

[2]A full comparison of related approaches is not undertaken here, as doing so would distract from the main philosophical aims of this paper, but, for the most similar approaches, compare the generalized approach to Cut Elimination put forward by Baaz, Fermüller, and Zach (1993) and the "unifying notation" for tableau systems, developed and deployed by Smullyan (1963, 1968 20-22) and Fitting (1983).

use in proofs, we should be intuitionists rather than classicalists, since it is intuitionistic natural deduction rather than classical natural deduction that displays that proof-theoretic virtue of *harmony*, with the introduction and elimination rules fitting together as they ought, with each set of rules being neither too strong nor too weak relative to the other.[3] In what is now a classic response to Dummett, Ian Rumfit (2000), drawing on prior work from Timothy Smiley (1996), shows that if one has a natural deduction system that contains not just rules for *affirming* sentences but rules for *denying* sentences as well, then it is easy to arrive at a harmonious natural deduction system for classical logic. The system Rumfitt proposes is thus *bilateral* in taking affirmation and denial as basic. In such a system, a well-formed formula must be prefaced with a positive or negative force-marker, expressing either affirmation or denial. The force-marker has wide scope over the whole sentence. Thus, the affirmation of a sentence φ might be written as $+\langle\varphi\rangle$, and the denial for φ can be written as $-\langle\varphi\rangle$. Unlike a negation operator, force-markers are neither embedable or iterable; there must always be exactly one force-marker and it must always be prefixed to a whole sentence. So, for instance, although both $+\langle p \wedge \neg q\rangle$ and $-\langle\neg p\rangle$ are well formed, neither $+\langle p \wedge -\langle q\rangle\rangle$ nor $-\langle-\langle p\rangle\rangle$ are well-formed.[4]

Now, the key innovation of Smiley/Rumfit style bilateral natural deduction systems are the rules for negation. In Rumfitt's system, they are the following:[5]

$$\frac{-\langle\varphi\rangle}{+\langle\neg\varphi\rangle}\ +_{\neg_I} \qquad\qquad \frac{+(\neg\varphi)}{-\langle\varphi\rangle}\ +_{\neg_E}$$

[3] I leave the notion of harmony informal in this preliminary presentation, as there are many differing conceptions of what, exactly, it amounts to.

[4] I find that the notation I use, where the unsigned sentence to which signs are attached are always enclosed in angle brackets makes this feature of the system more clear.

[5] Following secondary literature (e.g. Kurbis 2021), I will refer to system $\mathcal{B}$, originally put forward by Rumfitt (2000), as "Rumfitt's system," even though, in that article, Rumfitt expresses a preference for Smiley's pared-down system, which he calls $\mathcal{J}$.

$$\frac{+\langle\varphi\rangle}{-\langle\neg\varphi\rangle}\ -_{\neg I} \qquad\qquad \frac{-\langle\neg\varphi\rangle}{+\langle\varphi\rangle}\ -_{\neg E}$$

These rules jointly codify that denying a sentence has the same logical significance as asserting its negation. They are obviously harmonious, and they clearly define classical negation, as both double negation introduction and elimination are immediately through two applications of the I-rules or E-rules, respectively. The usual negation introduction rule is replaced by a generalized reductio principle, which, in a bilateral setting, is treated as a *structural rule*, not involving any specific logical connectives. Where A and B are signed formulas, and starring a formula yields the oppositely signed formula, the rule, which is known as "Smileian Reductio" (Rumfitt 2000, 804) can be put as follows:

$$\frac{\Gamma, A \vdash B \quad \Gamma, A \vdash B^*}{\Gamma \vdash A^*}$$

Just these negation rules, Smileian Reductio, along with the standard rules for conjunction, taken as the positive rules, itself constitutes a sound, complete, and harmonious system of classical propositional logic.[6] However, since Rumfitt's motivation is to provide a proof-theoretic analysis of the meanings of the logical connectives, the main theoretical fruit of the system is meant to be the rules he provides for the whole set of classical connectives.

Beyond the modification of the rules for negation, which crucially make use of the bilateral set-up, Rumfitt sticks to the standard rules from Gentzen as much as possible in his proposal of a bilateral proof system for classical logic. Exploiting the duality of conjunction and disjunction, the negative disjunction rules are of exactly the same form as the standard (positive) conjunction rules, replacing the positive signs left implicit in Gentzen's unilateral calculus with negative ones:

[6] It's straightforward (though a bit tedious) to derive the conjunction/negation translations of the standard Hilbert axioms, along with modus ponens from these rules.

$$\frac{-\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi \vee \psi\rangle}\,-_{\vee_I} \qquad \frac{-\langle\varphi \vee \psi\rangle}{-\langle\varphi\rangle}\,-_{\vee_{EL}} \qquad \frac{-\langle\varphi \vee \psi\rangle}{-\langle\psi\rangle}\,-_{\vee_{ER}}$$

Likewise, the negative conjunction rules are of exactly the same form as Gentzen's (positive) disjunction rules:[7]

$$\frac{-\langle\varphi\rangle}{-\langle\varphi \wedge \psi\rangle}\,-_{\wedge_{IL}} \qquad \frac{-\langle\psi\rangle}{-\langle\varphi \wedge \psi\rangle}\,-_{\wedge_{IR}} \qquad \frac{-\langle\varphi \wedge \psi\rangle \quad \begin{matrix}\overline{-\langle\varphi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{matrix} \quad \begin{matrix}\overline{-\langle\psi\rangle}^{\,v}\\ \vdots \\ \overline{A}\end{matrix}}{A}\,-_{\wedge_E}^{u,v}$$

The one set of connective rules that Rumfit cannot simply inherit from Gentzen are the negative conditional rules, since Gentzen himself does not provide rules for the dual of the conditional. Here, Rumfitt provides the following negative conditional rules:

$$\frac{+\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi \rightarrow \psi\rangle}\,-_{\rightarrow_I} \qquad \frac{-\langle\varphi \rightarrow \psi\rangle}{+\langle\varphi\rangle}\,-_{\rightarrow_{EL}} \qquad \frac{-\langle\varphi \rightarrow \psi\rangle}{-\langle\psi\rangle}\,-_{\rightarrow_{ER}}$$

Supplementing the (positive) rules for the binary connectives of NK with these negative rules yields a natural deduction system for classical logic that conforms to Dummett's demand of harmony. Thus, bilateralism promises to provide a way for the classical logician to adopt a proof-theoretic semantics for their preferred set of connectives. It is not, however, without worries. I'll restrict my attention to just two, concerning both *how many* rules there are in a bilateral setting and *which* bilateral rules count as definititive of the meanings of the connectives.

1.1 Worry One

Confronting Rumfitt's system for the first time, the first thing one might wonder is whether having *twice* the number of rules as Gentzen's NK is

[7]Note that, in keeping with the terminology used above, in the elimination rule here, "A" is an arbitrary *signed* formula.

too steep of a price to pay for harmony.[8] All else being equal, it seems that, if one is trying to proof-theoretically specify the meanings of the connectives in terms of proof rules, one should aspire to do this with the *minimal* number of rules that are sufficient for this specification. While it is surely not a devastating criticism of Rumfitt's bilateral proposal that it requires twice as many rules as familiar unilateral ones, it is certainly not a particularly welcome feature of the proposal.

Perhaps sensitive to this issue of rule proliferation, Smiley (1996) proposes a pared-down system with half the rules. However, he does so only by leaving out negative rules for conjunction and positive rules for disjunction, providing only rules for asserting conjunctions and rules for denying disjunctions. Thus, it seems less that we get a bilateral system with half the rules, and more that we get half a bilateral system. It seems clear that, when giving the rules for a given connective in a bilateral system, one ought to provide rules for both asserting and denying sentences with that connective. That's kind of the whole point of bilateralism.

It seems, then, that the bilateralist is simply committed to providing twice the rules. If the benefits of bilateralism are sufficiently great, then the added complexity can be a reasonable price to pay. But it's a price nonetheless.

1.2 Worry Two

Now, moving on from the number of rules to the rules themselves, the particular set of rules proposed by Rumfitt's is widely considered to be the definitive set of rules for bilateral classical logic.[9] However, as several authors have recently noted, there is reason to think that, once one goes bilateral, restricting oneself to the rule forms proposed by Gentzen is

[8]See, for instance, Restall (2020, 12) for an expression of this concern.

[9]By "widely" here, I mean to be speaking primarily of commentators other than the few who are serious bilateralists themselves. See, for instance, Restall (2020), Kurbis (2022).

somewhat arbitrary.[10] To see this, look again at the negative conditional rules above and see that they are notably different than all of the other binary connective rules in Rumfitt's system. In particular, they're the only binary connective rules in Rumfitt's system that are *bilaterally mixed* in the sense of involving both positive and negatively signed formulas. Once one sees that such rules are possible in bilateral system, it's natural to wonder why there's not more of them, and once one wonders this, one quickly sees that there is a number of possible harmonious bilateral proof systems with different rules for the connectives.

As an illustration of this point, compare the standard disjunction elimination rule found in Gentzen's LK, known as *argument by cases*:

$$\dfrac{\varphi \vee \psi \quad \begin{matrix} \overline{\varphi}^{\,u} \\ \vdots \\ \overline{\chi} \end{matrix} \quad \begin{matrix} \overline{\psi}^{\,v} \\ \vdots \\ \overline{\chi} \end{matrix}}{\chi} \vee_E^{u,v}$$

with the sort of disjunction elimination rule that most of us teach our students when we teach an introductory logic class, known as *disjunctive syllogism*:

$$\dfrac{\varphi \vee \psi \quad \neg\varphi}{\psi}$$

We teach this latter rule to our students because it is much more intuitive than the standard rule from Gentzen. Yet, as we all know, it's problematic from proof-theoretic perspective in two principal ways. First, this rule for the elimination of disjunction is not *separable* from the rules for negation; since it involves a negated formula, any system with these disjunction rules must also contain appropriate for negation. Second, and more fundamentally, this elimination rules is not *harmonious* with the standard set of introduction rules:

[10]See, for instance, Kurbis (2016), del Valle-Inclan and Julian Schlöder (2023)

$$\frac{\varphi}{\varphi \vee \psi} \vee_{I_L} \qquad \frac{\psi}{\varphi \vee \psi} \vee_{I_R}$$

If we're unilateralists, these two issues compel us to treat argument by cases as the main disjunction elimination rule; we simply have no alternative. If we're bilateralists, however, we have neither of these problems. First, rather than the above formulation of disjunctive syllogism in which negation figures, we have the following bilateral formulation:

$$\frac{+\langle\varphi \vee \psi\rangle \quad -\langle\varphi\rangle}{+\langle\psi\rangle} +_{\vee_E}$$

There is no issue of separability here, since the negative force-marker is treated as distinct from negation. Now, for the second problem, though this elimination rule is not harmonious with the standard disjunction introduction rules above, if we're bilateralists, these aren't our only option. With the possibility of bilaterally mixed rules at our disposal, we can formulate the following introduction rule which is harmonious with the above elimination rule:

$$\cfrac{\cfrac{\cfrac{}{-\langle\varphi\rangle}\, u \\ \vdots}{+\langle\psi\rangle}}{+\langle\varphi \vee \psi\rangle} +_{\vee_I}{}^{u}$$

These alternate rules for disjunction have recently been proposed by Pedro del Valle-Inclan and Julian Schlöder (2023), and they have every bit as much of a claim to being the bilateral rules definitive of the meaning of disjunction as those proposed by Rumfitt.[11]

So, if one adopts a bilateralist approach to proof-theoretic semantics, one has a great amount of theoretical freedom when it comes to determining the rules that one takes to be definitive of the meanings of the logical connectives. With theoretical freedom, however, comes theoretical

[11] Indeed, as I'll argue below, they have a better claim.

responsibility—in particular, the responsibility to provide a justification for one's theoretical choices. And it's not at all clear what the justificatory criteria that should be applied to settle the matter actually are. Whatever the justificatory criteria are, they are generally given the label "harmony," but the significance of this label, owed to Dummett, varies greatly from author to author. There are two dimensions of potential dispute here. The first is a familiar dispute over the criteria of *unilateral* harmony between introduction rules and elimination rules. Some authors require that harmony between introduction and elimination rules be ensured by the form of the elimination rules, requiring "generalized elimination" rules, whereas other authors, proposing more natural elimination rules, conceive of harmony in terms of a reduction and expansion procedure of the sort proposed by Pfenning and Davies (2005). Clearly, however, in a bilateral system, it's not just the *introduction* and *elimination* rules that need to be harmonious; the *positive* and *negative* rules should be harmonious as well. So, the bilateralist needs to supplement the criteria for unilateral harmony with some set of criteria for *bilateral* harmony. Here, there is even less clarity than in the more familiar context, with various offers proposing different criteria for bilateral harmony that the specific rules that they propose meet.[12] So, in the context of proof-theoretic semantics, bilateralism faces the challenge of specifying *which rules* define the meanings of the connectives. Though this challenge is not completely unique to bilateralism, as there are multiple possibilities for unilateral proof systems as well, given the multiplication of possible rule sets in a bilateral setting, this is a particularly pressing challenge for bilateralism.

[12]See, for instance, Francez (2014), Kurbis (2022), and del Valle-Inclan and Schlöder (2023).

2 A New Kind of Bilateral System

Bilateralism has principally been proposed as a way of formulating *natural deduction* systems, and it is in this context that I have formulated the above two worries. To respond to these worries, however, I want to introduce a new kind of bilateral system: a bilateral *sequent calculus* more in the spirit of Gentzen's LK than NK. There has been little to no development of bilateral sequent calculi along these lines.[13] It's not surprising that this is so. While the classical sequent calculus is quite nice, from a formal proof-theoretic perspective, its sequents feature *multiple conclusions*, and this is thought by many to disqualify it from being appealed to in the context of proof-theoretic semantics.[14] Natural deduction systems, in virtue of their "naturalness" (their close correspondence with ordinary reasoning practices), are widely thought to be preferable to sequent calculi for the purpose of proof-theoretic semantics, and bilateralism is generally proposed as a way of providing proof-theoretically virtuous natural deduction systems for classical logic. As I'll show, however, this is not all that bilateralism has to offer. Indeed, bilateralism has perhaps even more to offer in a sequent calculus setting, for a bilateral sequent calculus can avoid all of the worries associated with multiple conclusions sequent calculi while also resolving the above two worries with bilateralism. I'll now propose such a sequent calculus.

The proposed sequent calculus is a single conclusion calculus where each connective is given exactly two rules: a positive rule saying when one is to affirm a sentence with that main connective and a negative rule saying when one is to deny a sentence with that main connective. A sequent of the form $\Gamma \vdash A$, where Γ is a set of signed formulas and A is a single signed formula, can be read as saying that taking all of the stances in Γ, be they affirmations or denials, commits one to taking the stance A,

[13]One exception is Ayhan's (2021) development of a bilateral intuitionistic sequent calculus, following Wansing (2016).

[14]See, for instance, Rumfitt (2008) and Steinberger (2011).

be it an affirmation or denial, where the notion of commitment here along the lines developed by Brandom (1994).[15] The sole axiom schema is that of Containment (or Contexted Reflexivity):

$$\frac{}{\Gamma, A \vdash A}\ \text{CO}$$

Where Γ and $\{A\}$ contain only signed atomics.

The one substantive structural rule that is necessary for the sequent calculus to function is the generalized contraposition principle that Smiley (1996) dubs *Reversal*. Where A and B are signed formulas, and starring a signed formula yields the oppositely signed formula, the principle can be put as follows:

$$\frac{\Gamma, A \vdash B}{\Gamma, B^* \vdash A^*}\ \text{RV}$$

Where {A} or {B} can be null.

There are two sorts of instances of Reversal that are of particular note, conceptually.

The first sort of instance of Reversal worth noting is where A and B are of opposite signs, that is, instances of Reversal of one of the following two forms:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma, +\langle\psi\rangle \vdash -\langle\varphi\rangle}\ \text{RV} \qquad \frac{\Gamma, -\langle\varphi\rangle \vdash +\langle\psi\rangle}{\Gamma, -\langle\psi\rangle \vdash +\langle\varphi\rangle}\ \text{RV}$$

In the first sort of case, Reversal tells us that if, relative to a context Γ, affirming φ commits one to denying ψ, then affirming ψ commits one to denying φ. Now, if the affirmation one sentence commits one to denying the other, then these sentences can be said to be *incompatible*. So, in this case, Reversal tells us that the relation of incompatibility between sentences (relative to a context Γ) is *symmetric*. Alternately, in the second

[15]Note, it is just for simplicity's sake that I am here taking what occurs on the left of a sequent to be a *set* of signed formulas rather than a list or a multi-set.

sort of case, Reversal tells us that if, relative to a context Γ, denying φ commits one to affirming ψ, then denying ψ commits one to affirming φ. So, in addition to encoding the symmetry of incompatibility or (to put in Aristotelian vocabulary) *contraeity*, Reversal also encodes the symmetry of *subcontraeity*.

The second sort of instance of Reversal worth noting is where $\{A\}$ or $\{B\}$ is null. This is not permitted in Smiley's formulation of the rule, but it makes sense in the context of formulating a bilateral sequent calculus, permitting the introduction of sequents with empty right-hand sides. In a standard unilateral sequent calculus, a sequent with a set of sentences on the left side of the turnstile and an empty right side encodes the *incoherence* of that set of sentences. Thus, for instance, the sequent $p, q \vdash$ can be read as saying that the set $\{p, q\}$ is incoherent or that p and q are incompatible, as codified by the fact that the negation rules let you move from $p, q \vdash$ to $p \vdash \neg q$ and $p \vdash \neg q$. In this bilateral system, we get the same behavior at the structural level by the following two instances of Reversal, where A and B are null:

$$\frac{\Gamma \vdash A}{\Gamma, A^* \vdash}\ \text{RV} \qquad\qquad \frac{\Gamma, A \vdash}{\Gamma \vdash A^*}\ \text{RV}$$

The first instance says that if, Γ commits one to taking stance A, then the set of stances consisting in Γ along with the opposite stance of A is incoherent. The second instance says that if, Γ along with stance A constitutes an incoherent set of stances, then Γ commits one to the opposite stance of A. Though one might object to these principles in some contexts, I take them to be partly constitutive of a classical understanding of consequence and incoherence.[16]

[16] Ripley (2013), for instance, would presumably reject the second instance, since he would want to say that any position along with affirming the liar sentence is incoherent, but he doesn't thereby want to say that any position commits one to denying the liar sentence, since he thinks that set of stances too is incoherent. Developments of this system absent Reversal to accommodate this thought are possible but not explored here.

With the conceptual significance of Reversal explicated, we can note that its technical significance is that it enables us to put forward a system with rules for introducing positively and negatively signed formulas on just one side of the turnstile, since one can get a signed formula with a given connective on the other side of the turnstile by getting its opposite on the side for which rules are given and using Reversal.[17] Following the paradigm of privileging *introduction* rules in the context of proof-theoretic semantics, the main system proposed here contains solely right rules.

Now, rather than introducing rules for particular connectives, I will introduce rules for connectives by way of general rule *schemas*. To do this, I deploy a notation that schematizes over signs, using variables such as $\boldsymbol{a}$ and $\boldsymbol{b}$ to indicate signs that may be either $+$ or $-$, where starring a variable changes the sign it is assigned to its opposite. Officially, the star is a function defined over $\{+, -\}$, mapping $+$ to $-$ and $-$ to $+$. So, for any signed formula of the form $\boldsymbol{a}\langle\varphi\rangle$, where $\boldsymbol{a} \in \{+, -\}$, if $\boldsymbol{a} = +$ then $\boldsymbol{a}^* = -$, and if $\boldsymbol{a} = -$ then $\boldsymbol{a}^* = +$ (and so $\boldsymbol{a}^{**} = \boldsymbol{a}$).[18] My main concern here is with the schema for the binary connectives, but, to be consistent, I will deploy

[17]A similar idea is invoked by Smiley (1996) in the pared-down system mentioned above. It's worth noting that the treatment of Reversal here goes in the face of a recent argument from del Valle Inclan and Schlöder that bilateral structural rules—often called "co-ordination principles" (Rumfitt 2000, 804)—should be restricted to atomic sentences. Their argument involves considering bilateral tonk-like connectives that trivialize the consequence relation if certain co-ordination principles, such as Smileian Reductio (considered below), are not limited to atomic sentences. By requiring that any proof in which the such co-ordination principles are used on complex sentences can be transformed into one in which they are used only on atomics, del Valle Inclan and Schlöder rule out these tonk-like connectives. This is certainly a good argument for the claim that a bilateral system ought to be such that co-ordination principles like *Smileian Reductio* can be restricted to atomics (or, as I will show for the system here, eliminated entirely). However, the reasoning here does not extend to *Reversal* which is not a simplifying rule like Smileian Reductio, and so poses no risk of trivialization when used with disharmonious connectives. In effect, Reversal is treated here in a way that is more analogous to how Exchange is treated in a standard unilateral sequent calculus (where sequents are treated as relating lists rather than sets), whereas Smiliean Reductio is treated as analogous to a structural rule like Cut.

[18]So that the star is not ambiguous, we might now say that, where A is shorthand for a formula of the form $\boldsymbol{a}\langle\varphi\rangle$, A^* is shorthand for $\boldsymbol{a}^*\langle\varphi\rangle$.

this strategy for the sole unary connective as well. Where $\triangleright$ is any unary connective, the rule schema is the following:

$$\frac{\Gamma \vdash \boldsymbol{a}^*\langle\varphi\rangle}{\Gamma \vdash \boldsymbol{a}\langle\triangleright\varphi\rangle}\ \boldsymbol{a}_\triangleright \qquad\qquad \frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle}{\Gamma \vdash \boldsymbol{a}^*\langle\triangleright\varphi\rangle}\ \boldsymbol{a}^*{}_\triangleright$$

Clearly, whether $\boldsymbol{a}$ is $+$ or $-$, there is only one set of rules defined by this schema, and it is the familar set of positive and negative rules for negation:

$$\frac{\Gamma \vdash -\langle\varphi\rangle}{\Gamma \vdash +\langle\neg\varphi\rangle}\ +_\neg \qquad\qquad \frac{\Gamma \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\neg\varphi\rangle}\ -_\neg$$

To these negation rules, we add the following general schema for the binary connective rules, where $\circ$ is any binary connective:

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_\circ \qquad\qquad \frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ$$

The $\boldsymbol{c}_\circ$ rule says that if Γ commits one to taking stance $\boldsymbol{a}$ to φ and Γ also commits one to taking stance $\boldsymbol{b}$ to ψ, then Γ commits one taking stance $\boldsymbol{c}$ to $\varphi \circ \psi$. The $\boldsymbol{c}^*{}_\circ$ rule says that if Γ along with the stance $\boldsymbol{a}$ to φ commits one to taking $\boldsymbol{b}^*$, the opposite of stance $\boldsymbol{b}$, to ψ, then Γ commits one to taking $\boldsymbol{c}^*$, the opposite of stance $\boldsymbol{c}$, to $\varphi \circ \psi$. To understand the conceptual significance of the $\boldsymbol{c}^*{}_\circ$ rule, note first that, given Reversal, the premise sequent is immediately interprovable with $\Gamma, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}^*\langle\varphi\rangle$. Thus, the $\boldsymbol{c}^*{}_\circ$ rule can be understood as saying that if, relative to Γ, the stances $\boldsymbol{a}\langle\varphi\rangle$ and $\boldsymbol{b}\langle\psi\rangle$ are *incompatible* in the sense that, if one takes one of these stances, one is committed to taking the opposite of the other, then Γ commits one to taking $\boldsymbol{c}^*$, to $\varphi \circ \psi$. Even at this very high level of abstraction, we can make sense, conceptually, of how rules of these two forms fit together as rules for opposite stances towards a sentence should. Concretely, we can prove that they fit together harmoniously by proving a general bilateral analogue to Cut Elimination that shows that the rule that I'll call *Bilateral Reductio (BR)*:

$$\frac{\Gamma \vdash A \qquad \Delta \vdash A^*}{\Gamma, \Delta \vdash}\ \mathrm{BR}$$

is admissible with respect to rules of this form. The proof, provided in the Appendix, proceeds analogously to standard Cut Elimination for multiple conclusion sequent calculi but at this higher level of generality.

Bilateral Reductio says that if the position Γ commits one to the stance A, and the position Δ commits one to the opposite stance A^*, then the position Γ, Δ is incoherent. The eliminability of BR in a bilateral sequent system means that this is always already the case, and so BR is never actually needed. To get a sense of the conceptual significance of the proof of the eliminability of BR in the schematic sequent system proposed here consider that it's clear, given the intuitive description of these rules above, that if a position Γ commits one to one stance towards $\varphi \circ \psi$ and a position Δ commits one to the opposite stance towards $\varphi \circ \psi$, then Γ, Δ is already incoherent in virtue of committing one to opposite stances to the simpler formulas in $\varphi \circ \psi$. We might show this by the following transformation:[19]

$$\frac{\dfrac{\dfrac{\vdots^{\,n}}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle} \qquad \dfrac{\vdots^{\,m}}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_\circ \qquad \dfrac{\dfrac{\vdots^{\,k}}{\Delta, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}}{\Delta \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_\circ}{\Gamma, \Delta \vdash}\ \mathrm{BR}$$

$$\rightsquigarrow$$

$$\frac{\dfrac{\vdots^{\,n}}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle} \qquad \dfrac{\dfrac{\dfrac{\vdots^{\,m}}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle} \qquad \dfrac{\vdots^{\,k}}{\Delta, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}}{\Gamma, \Delta, \boldsymbol{a}\langle\varphi\rangle \vdash}\ \mathrm{BR}}{\Gamma, \Delta \vdash \boldsymbol{a}^*\langle\varphi\rangle}\ \mathrm{RV}}{\Gamma, \Delta \vdash}\ \mathrm{BR}$$

This is the crucial case of the BR-elimination proof provided in the Appendix, where a proof in which BR has been applied to premises express-

[19]Note, once again, that I am treating what goes on the left side of the turnstile as a *set* of signed formulas, and so there is no need to apply Contraction in the final step of the transformed proof.

ing commitments to opposite stances towards $\varphi \circ \psi$ is transformed to one in which it has been applied to premises expressing commitments to simpler signed formulas. The fact that such a reduction in the complexity of formulas to which BR is applied is possible means that the introduction rules for $\circ$ do not enable one to introduce opposite stances towards $\varphi \circ \psi$ in any case in which one's stances do not already commit one to opposite stances towards simpler sentences. In general, proving the eliminability of BR can be understood as showing that connective rules of this form do not generate incoherence when it is not already there. In any case in which we apply BR after applying the positive and/or negative rules yielded by our connective schema, concluding the incoherence of a position, we could have just as well applied BR prior to applying these rules, and so that position must have already been incoherent prior to the application of the connective rules.

The proof of the eliminability of BR in this sort of bilateral system is an analogue to the proof of Cut Elimination in a standard unilateral sequent system, and several familiar and important facts follow from the proof. First, it follows that a logic L consisting in rules of this form is consistent in the sense that, if $\vdash_L A$, then $\nvdash_L A^*$. It also follows that introducing any connective into a language with rules of this form to an existing consequence relation $\vdash$ will constitute a *conservative extension* of that langauge. Thus, no connective that is introduced in accordance with this schema will produce tonk-like behavior, functioning as a "runabout inference-ticket," (Prior 1960). Further, the admissibility of other, more familiar structural rules follows from the eliminability of Bilateral Reductio. First, it follows directly that the more familiar (to bilateralists) structural rule of *Smilean Reductio*, shown in Section 1 above, is admissible. Second, it follows directly that the more familiar (to non-bilateral logicians) *Cut* Rule:

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}$$

is likewise admissible. What is wonderful about this sort of system is that once one proves these results at the general schematic level, one has ipso facto proven these results for whatever connectives one introduces through instances of this schema. In general, a bilateral system of this sort enables one to do one's metatheory at a higher level of generality, greatly simplifying the proofs. It's not, however, simply a technical trick—as I hope to have made clear, metatheoretic results at the schematic level of generality actually make good intuitive sense.

Simply by varying the signs assigned to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, one arrives at rules for all of the classical connectives. Moreover, when the rules for a given connective have been provided in accordance with this schema, the rules for its dual can be obtained simply by taking the opposite of all the signs.[20] The instance of this general schema that is most immediately obvious are the rules for the conditional:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash +\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \to \psi\rangle}\ +_{\to} \qquad\qquad \frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \to \psi\rangle}\ -_{\to}$$

These are simply the introduction rules provided in Rumfitt's (2000) natural deduction system. However, sequent rules for conjunction and disjunction following exactly the same schema can be given as follows:[21]

$$\frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash +\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \wedge \psi\rangle}\ +_{\wedge} \qquad\qquad \frac{\Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \wedge \psi\rangle}\ -_{\wedge}$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash +\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \vee \psi\rangle}\ +_{\vee} \qquad\qquad \frac{\Gamma \vdash -\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \vee \psi\rangle}\ -_{\vee}$$

The system consisting in these bilateral connective rules, with the axiom schema of Containment and the structural rule of Reversal (and the

[20]The dual C^d of an n-ary connective C is defined such that $\neg C(\varphi_1, \varphi_2 \ldots \varphi_n) \equiv C^d(\neg\varphi_1, \neg\varphi_2 \ldots \neg\varphi_n)$. It should be clear given this definition why reversing all of the signs in the rules for a given connective yields the rules for its dual.

[21]Conjunction and disjunction rules of this form have recently been proposed by del Valle Inclan and Schloder (2023).

structural rules that entitle us to treat the formulas on the left side of a sequent as a set), is a sound and complete system of classical logic. In fact, as I show in the Appendix, it's equivalent to Ketonen's (1944) multiple conclusion classical sequent calculus, which has several nice formal properties all of which are preserved in this system.[22] Like Ketonon's calculus, proof of a sequent is achieved through direct decomposition of that sequent, starting with the sequent one wants to prove at the "root" of a proof tree, and working one's way up through application of whatever rules one can apply, using only the structural rule of Reversal. However, whereas Ketonon's system is essentially multiple conclusion, this bilateral system uses only single conclusion sequents, and thus provides a more immediately intuitive explication of the sense of the connectives in terms of the conditions under which one is to affirm or deny a sentence with that connective.[23]

In addition to the standard classical connectives, the same general rule schema also yields rules for the Sheffer Stroke and Pierce's Arrow:[24]

$$\frac{\Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \mid \psi\rangle}\ +_{\mid} \qquad\qquad \frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash +\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \mid \psi\rangle}\ -_{\mid}$$

$$\frac{\Gamma \vdash -\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \downarrow \psi\rangle}\ +_{\downarrow} \qquad\qquad \frac{\Gamma, -\langle\varphi\rangle \vdash +\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \downarrow \psi\rangle}\ -_{\downarrow}$$

The pair of rules for either of these connectives alone constitutes a sound and complete system of classical logic. These rules provide as good of a proof-theoretic explication of the sense of the connectives as any. In particular, they make it clear that they needn't be thought of as conceptually posterior to the standard connectives such as $\wedge$ and $\neg$. Finally, note that

[22]The Ketonen system to which this system is equivalent has the same rules as the system Negri and von Plato (2008) call "G3cp," but with the standard negation rules of LK.

[23]See, for instance, Rumfit (2008) and Steinberger (2011) for relevant criticisms of multiple conclusion sequent calculi.

[24]These rules are equivalent to the multiple conclusion sequent rules provided by Riser (1967) and Zach (2016).

the schema yields the following rules for $>\!\!-$, the oft-forgotten dual of $\to$:[25]

$$\frac{\Gamma \vdash -\langle \varphi \rangle \quad \Gamma \vdash +\langle \psi \rangle}{\Gamma \vdash +\langle \varphi >\!\!- \psi \rangle} +_{>\!-} \qquad \frac{\Gamma, -\langle \varphi \rangle \vdash -\langle \psi \rangle}{\Gamma \vdash -\langle \varphi >\!\!- \psi \rangle} +_{>\!-}$$

And, of course, flipping the signs of the premise sequents in the rules for $\to$ and $>\!\!-$, we get the rules for $\leftarrow$ and $-\!\!<$.

So, in all, we have the following assignments of signs for the binary connectives:

$$\begin{array}{cc}
\wedge\colon \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = + & \vee\colon \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = - \\
|\colon \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = - & \downarrow\colon \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = + \\
\to\colon \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = - & >\!\!-\colon \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = + \\
-\!\!<\colon \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = + & \leftarrow\colon \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = -
\end{array}$$

Thus, the only non-degenerate binary connectives that the schema doesn't yield are the exclusive disjunction and the biconditional.[26] At the end of this paper, I will return to answer the question of why, exactly, this schema yields the rules for just the set of connectives that it does. First, however, I want to explain how this system responds to the two worries voiced above.

3 Responding to the Two Worries

I call the system constituted by all of the connective rules directly given by the schema—every permutation of positive/negative signs assigned to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in the schema above—"BK," for "Bilateral Ketonen-style." Though, technically, BK contains all of the binary connectives definable

[25]See Wansing (2016), Kurbis (2019, 256), and Ayhan for discussion of this connective in a bilateralist intuitionistic context. It's perhaps worth noting that Wansing and Ayhan provides rules for $-\!\!<$, which technically is the dual of $\leftarrow$ rather than $\to$.

[26]By "degenerate binary connectives," I mean the ones such that $\varphi \circ \psi \equiv \varphi$ or $\varphi \circ \psi \equiv \psi$.

by the rule schema, one can take whatever fragment of BK consisting of whatever rules one like. I claim that BK constitutes a response to the two worries voiced above. Let me now defend this claim.

3.1 Responding to Worry Number One

It is clear how system BK responds to the first worry—that going bilateral involves specifying too many rules. In the proposed sequent calculus, there are exactly two rules for each binary connective: a rule that says when one is to affirm a sentence with that main connective and a rule that says when one is to deny a sentence with that main connective. The rules yielded by this schema for each connective are proposed as definitive of the meaning of that connective, and, with only two rules for each connective, this is as minimal of a set of rules as has been proposed by anyone in proof-theoretic semantics. One might worry that, like Smiley's (1996) pared-down system, the proposed sequent system is incomplete, not from a technical perspective but from a conceptual perspective. Clearly, however, this calculus does not have the shortcoming that Smiley's has of providing rules only for affirming some connectives and rules only for denying others; each connective gets one rule for affirming and one rule for denying. Now, one might worry that providing only *introduction* rules (right rules, in the context of a sequent system) and not *elimination* rules (left rules) leaves the system conceptually incomplete. However, there is a strong tradition in proof-theoretic semantics, going back to Gentzen (1935) himself, that the introduction rules should be given pride of place in such a semantics:

> "The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions," (80).

Following this line of Gentzen's, I take it that these rules are sufficient to serve as “definitions” of the connectives.

Though I follow Gentzen in privileging introduction rules as definitive of meaning, the introduction rules proposed here have a crucial feature that elevates their status, in the context of proof-theoretic semantics, above the rules that Gentzen himself proposes: they are all *invertible* in that whenever a sequent involving a sentence featuring some connective is derivable, the premise sequents involving the connected sentences are also derivable. Thus, the premise sequents specify *exactly* the conditions under which one is committed to affirming or denying a compound sentence. This licenses us to say, for instance, that one is committed to affirming a conjunction *just in case* one is committed to affirming both of the conjuncts, and one is committed to denying a conjunction *just in case* affirming one of the conjuncts commits one to denying the other. This seems to me to be a perfectly sufficient specification of the meaning of conjunction, formally encoded with just two invertible proof rules. Similar things can be said for all of the other connective rules.

So, there is good reason to think that these rules themselves are sufficient for proof-theoretically specifying the meanings of the classical connectives. Nevertheless, though one doesn’t *need* more rules for the purpose of proof-theoretic semantics, if one *wants* more rules, one can have them. Though the rules yielded by the schema are treated here principally as sequent rules which themselves constitutes a complete logical system, the schema can also be treated as a schema for introduction rules in the context of a natural deduction system, and schematic elimination rules can be straightforwardly extracted from them. For instance, simply exploiting the invertability of the rules, one arrives at a schematized version of the proposed system of del Valle Inclan and Schlöder (2023), resulting from supplementing the above schema with the following elimination rules:[27]

[27]For continuity with the main proposal, I present the natural deduction rules in sequent notation here. For how to translate into the more familiar notation for natural deduction, see Francez (2015, 16-28). Note that exactly which coordination principles

$$\frac{\Gamma \vdash \boldsymbol{c}\langle \varphi \circ \psi \rangle}{\Gamma \vdash \boldsymbol{a}\langle \varphi \rangle}\ \boldsymbol{c}_{\circ_{E_L}} \qquad \frac{\Gamma \vdash \boldsymbol{c}\langle \varphi \circ \psi \rangle}{\Gamma \vdash \boldsymbol{b}\langle \psi \rangle}\ \boldsymbol{c}_{\circ_{E_R}} \qquad \frac{\Gamma \vdash \boldsymbol{c}^*\langle \varphi \circ \psi \rangle \quad \Gamma \vdash \boldsymbol{a}\langle \varphi \rangle}{\Gamma \vdash \boldsymbol{b}^*\langle \psi \rangle}\ \boldsymbol{c}^*{}_{\circ_E}$$

Alternately, than these elimination, one can supplement the schematic introduction rules with "generalized elimination rules" (Schroder-Heister 1984) of the sort proposed in this context by Frances (2014) and Kurbis (2022):

$$\frac{\Gamma \vdash \boldsymbol{c}\langle \varphi \circ \psi \rangle \quad \Gamma, \boldsymbol{a}\langle \varphi \rangle, \boldsymbol{b}\langle \varphi \rangle \vdash A}{\Gamma \vdash A}\ \boldsymbol{c}_{\circ_E} \qquad \frac{\Gamma \vdash \boldsymbol{c}^*\langle \varphi \circ \psi \rangle \quad \Gamma \vdash \boldsymbol{a}\langle \varphi \rangle \quad \Gamma, \boldsymbol{b}^*\langle \psi \rangle \vdash A}{\Gamma \vdash A}\ \boldsymbol{c}^*{}_{\circ_E}$$

There other different possibilities here, and it's not my intention to settle them. Indeed, it's one of the important upshots of proposing a sequent system, as I have done here, that one *doesn't need to* settle these possibilities, since the system is complete with just introduction rules and it is strictly speaking only the introduction rules that are proposed as definitive of the meanings of the connectives. Nevertheless, it's worth noting here in the context of responding to Worry One that even if one *does* think it's essential to adopt a natural deduction system with both introduction and elimination rules to provide a proof-theoretic semantics for the logical connectives, and so the particular system one proposes will have twice the rules as standard unilateral natural deduction systems, the charge about the proliferation of rules is still reasonably parried here, since one only needs to specify the general schema to yield all of the rules.

3.2 Responding to Worry Number Two

Let us now turn to the second worry, that bilateral systems allow for too much freedom, making it unclear which set of bilateral proof rules is to

one wants in one's natural deduction system is a matter of possible dispute (for instance, it is natural to take the stronger Smilean Reductio rather than Reversal as basic in a natural deduction context). For discussion of coordination principles in the context of this system, see del Valle Inclan and Schlöder (2023) and del Valle Inclan (forthcoming). An open question is how to extend the treatment of material inferences discussed in Section 3.2 below, which is seamless in this sequent setting, to the natural deduction setting.

actually count as definitive of the meanings of the logical connectives. I claim that BK responds to this worry as well, as it has a reasonable claim to a *privileged* place among bilateral systems for classical logic.

The first point to make in this regard, which I hope to have illustrated at this point, is that the conceptual and technical benefits of having a single schema yield the rules for all of the connectives seem to be sufficiently great as to rule out any proposed set of bilateral connective rules that are *not* yielded by a single schema in this way. This rules out nearly all of the existing rule sets proposed in the literature. In particular, nearly every system proposes different rule forms for conjunction and disjunction, on the one hand, and the conditional, on the other. Of course, in the context of non-classical logics such as intuitionistic or relevant logics, there is reason for an asymmetry here. In the context of *classical* logic, however, there is no reason for such an asymmetry. The symmetry of all of the connectives is, indeed, a crucial aspect of their classicality and it makes good sense for this symmetry to be reflected in the rules that are taken to be definitive of their meanings.

So, there is good reason to think of the rules for the classical connectives as defined by way of a rule schema. The question, then, becomes whether the schema proposed is the only possible one. In our schema, of which Rumfitt's rules for the conditional are an instance, each connective gets, in the terminology of Francez (2014), a single categorical combining rule and a single hypothetical rule. One might instead generalize the form of Rumfitt's conjunction and disjunction rules, where each connective is given one categorical combining rule and two categorical *splitting* rules, with the following schema:

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_{\circ} \qquad \frac{\Gamma \vdash \boldsymbol{a}^{*}\langle\varphi\rangle}{\Gamma \vdash \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^{*}{}_{\circ_L} \qquad \frac{\Gamma \vdash \boldsymbol{b}^{*}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^{*}{}_{\circ_R}$$

Rules of this form correspond more closely to the rules of Gentzen's LK (with the exception of his conditional rules), whereas, as we've already pointed out, our rules correspond to those of Ketonen's classical sequent calculus. Such rules might seem more natural to many. However, there are several reasons to think that the complement to the categorical combining rule should be the hypothetical rule of the schema proposed here rather than the two splitting rules that one gets in standard bilateral systems. Very quickly, here are just four of them:

1. Though, given Reversal, the system with Gentzen-style rules is technically complete (with the addition of Contraction), when it comes to providing a proof-theoretic account of the meanings, the naturalness of proofs is a factor (though not a decisive one), and the system with Ketonen-style rules is much nicer. For example, compare the proofs of the law of excluded middle in the two systems:

$$\cfrac{\cfrac{-\langle p\rangle \vdash -\langle p\rangle}{-\langle p\rangle \vdash +\langle \neg p\rangle}\;{+_{\neg}}}{\vdash +\langle p \vee \neg p\rangle}\;{+_{\vee}}
\qquad\qquad
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{+\langle p\rangle \vdash +\langle p\rangle}{+\langle p\rangle \vdash +\langle p \vee \neg p\rangle}\;{+_{\vee_L}}}{-\langle p \vee \neg p\rangle \vdash -\langle p\rangle}\;{\text{RV}}}{-\langle p \vee \neg p\rangle \vdash +\langle \neg p\rangle}\;{+_{\neg}}}{-\langle p \vee \neg p\rangle \vdash +\langle p \vee \neg p\rangle}\;{+_{\vee_L}}}{-\langle p \vee \neg p\rangle, -\langle p \vee \neg p\rangle \vdash}\;{\text{RV}}}{-\langle p \vee \neg p\rangle \vdash}\;{\text{Cnt}}}{\vdash +\langle p \vee \neg p\rangle}\;{\text{RV}}$$

2. As, as we've already discussed, the Ketonen-style rules are *invertible* whereas the Gentzen-style rules are not, and this is not only technically advantageous but also conceptually significant in the context of providing a proof-theoretic semantics. In particular, the Ketonen-style hypothetical rules give us necessary and sufficient conditions for affirming or denying some logically complex sentence (enabling us to say, for instance that one is committed to affirming a disjunction *just in case* denying one of the disjuncts commits one to affirming the other), whereas the Gentzen-style splitting rules give us two distinct sufficient conditions for when one is committed to affirming or denying a logically complex sentence, neither of which is necessary.

3. It seems clear that the notion of *conditional proof*—one of the key innovations of Gentzen's proof systems—ought to figure at the very

least in our understanding of the conditional. The Gentzen-style conditional rules define the conditional in such a way that the sense of "If . . . then . . ." seems completely lost. Once we make this point for the conditional, and we see that there are equally intuitive introduction rules for conjunction and disjunction involving conditional proof, there is reason to think that the notion of conditional proof should figure in our understanding of all of the logical connectives.

4. Though the structural rule of Weakening is technically eliminable in a sequent calculus with Gentzen-style rules and the Ketonen-style axiom schema, one can't use such a logic to extend a consequence relation in which Weakening actually fails, since the Gentzen style rules enforce weakening with connectives. The Ketonen-style rules, on the other hand, do not have this issue.

I take it that, taken collectively, these reasons are pretty decisive. However, the last reason, I believe, is the most significant, and it requires some explanation.

I have laid out this bilateral sequent calculus in the context of a system in which the only axioms are sequents of the form $\Gamma, A \vdash A$. As a result, the only consequences and theorems are logical ones. Clearly, however, our reasoning with the use of logical connectives—which accounts for our understanding of their meaning—is not restricted to purely logical contexts, where the only things we ever affirm or deny are tautologies or contradictions. We also make affirmations and denials about such things as colors and shapes, animals, and other people, using logical words like "not," "and," and "or" in our making of affirmations and denials about such things. It seems that our proof-theoretic semantics for the logical connectives ought to be able to accommodate this fact, and it can if we can extend our logical system to permit the addition of non-logical *material* axioms such as the following:[28]

$+\langle \textbf{red} \rangle \vdash +\langle \textbf{colored} \rangle$

[28]This approach follows the approach to (unilateral) sequent calculi laid out by Hlobil (2016), Kaplan (2018), and Brandom (2018).

$+\langle\textbf{blue}\rangle \vdash +\langle\textbf{colored}\rangle$
$+\langle\textbf{mammal}\rangle \vdash -\langle\textbf{lays eggs}\rangle.$

From these non-logical axioms, we get, via the logical rules of both sequent calculi, such sequents as $+\langle\textbf{red} \vee \textbf{blue}\rangle \vdash +\langle\textbf{colored}\rangle$ and $+\langle\textbf{mammal}\rangle \vdash +\langle\neg\textbf{lays eggs}\rangle$. As this last sequent illustrates, however, many of the material inferences we make about such things as animals are *defeasible*, and so the structural rule of Weakening fails for the sequents codifying those inferences. For instance, though $+\langle\textbf{mammal}\rangle \vdash -\langle\textbf{lays eggs}\rangle$, it's not the case that $+\langle\textbf{mammal}\rangle, +\langle\textbf{platypus}\rangle \vdash -\langle\textbf{lays eggs}\rangle$. This is no problem for the bilateral Kenton-style rules, since the structural rule of Weakening is eliminable and the connective rules do not enforce Weakening with conjuncts or disjuncts. The Gentzen-style sequent calculus, however, has no way to accommodate such failures of Weakening.

Consider, for instance, that on the Gentzen-style schema, we have the following negative conjunction rule:

$$\frac{\Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \wedge \psi\rangle}$$

When we allow non-logical material axioms that codify defeasible material inferences, this rule lets us infer as follows:

$$\frac{+\langle\textbf{mammal}\rangle \vdash -\langle\textbf{lays eggs}\rangle}{+\langle\textbf{mammal}\rangle \vdash -\langle\textbf{platypus} \wedge \textbf{lays eggs}\rangle}$$

But this isn't a good inference. Affirming to "x is a mammal" commits one to denying "x lays eggs," but affirming "x is a mammal" doesn't commit one to denying "x is a platypus and x lays eggs." On the contrary, affirming "x is a mammal" is perfectly consistent with affirming "x is a platypus and x lays eggs." This problem is avoided with our negative conjunction rule:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \wedge \psi\rangle}$$

This rule precludes one from being able to derive the sequent $+\langle \textbf{mammal} \rangle \vdash -\langle \textbf{platypus} \wedge \textbf{lays eggs} \rangle$, since, given the set of material axioms corresponding to our ordinary material inferences about animals, though one has the material axiom $+\langle \textbf{mammal} \rangle \vdash -\langle \textbf{lays eggs} \rangle$, one *doesn't* have the material axiom $+\langle \textbf{mammal} \rangle, +\langle \textbf{platypus} \rangle \vdash -\langle \textbf{lays eggs} \rangle$. Since one needs this latter sequent in order to apply the negative conjunction rule, one can't derive the sequent saying that affirming "*x* is a mammal" commits one to denying "*x* is a platypus and *x* lays eggs." A full exposition of how BK can function to extend a set of material axioms codifying defeasible inferences is beyond the scope of the current paper, but I hope I've said enough to show that the ability to play nicely with such material axioms is a crucial advantage of the Ketonen-style bilateral rules that I have laid out here.[29]

To sum up, insofar as one is putting forward a bilateral system for the classical connectives, there is good reason to put forward a system in which the rules are derived by a single rule schema, as I have done here. Moreover, the schema for bilateral rules that I've articulated here has several crucial advantages over the only other plausible alternative, the main advantage being that these rules enable one to think about the logical connectives as used in defeasible reasoning, and this is critical when thinking about the meanings of the connectives grasped by speakers of a natural language such as English. There is good reason to think, then, that the set of rules yielded by the schema I've put forward here really is

[29] It's worth noting that, if one does extend a system to include material axioms encoding defeasible inferences, then it's not just Weakening that one must drop—principles such as Transitivity and Bilateral Reductio must be dropped for the same reasons. For an intuitive failure of BR closely related to the failures of Weakening discussed here, consider that affirming "*x* is a mammal" commits one to denying "*x* lays eggs" and affirming "*x* is a platypus" commits one to affirming "*x* lays eggs," but affirming "*x* is a mammal" and affirming "*x* is a platypus" is not (even defeasibly) incoherent. Thus, in addition to the reasons cited above, it is of crucial importance that BR can be eliminated from the proof of any classically valid sequent, since might want to extend classical logic with sequents for which it actually fails. (A similar idea, though with different kinds of cases (semantic paradoxes), is explored by Ripley (2013).)

definitive of the meanings of the classical connectives.

4 Explaining the Rule Schema

The question remains of why, exactly, all the connective rules can be derived from a single bilateral schema in the way we've done here. An answer, drawn from Zach (2015, 2020), involves thinking of the rules as generated from the normal forms of the connectives.[30] Note that the set of connectives yielded by this schema are just those for which there is a distinguished row of the truth-table (one F among all Ts or one T among all Fs). This means that they are just those for which pair $\langle p \circ q, \neg(p \circ q)\rangle$ is such that one member of the pair can be put into normal form with exactly two conjuncts of literals (specifying the distinguished row), and the other member can be put into normal form with exactly two disjuncts of literals (specifying all the other rows).[31] Expressing these normal forms without the use of negation, they are the following:

> $\varphi \wedge \psi$ is true (i.e. is to be affirmed) iff φ is true (i.e. is to be affirmed) and B is true (i.e. is to be affirmed).
>
> $\varphi \wedge \psi$ is false (i.e. is to be denied) iff φ is false or ψ is false (i.e. affirming one commits one to denying the other).
>
> $\varphi \vee \psi$ is true (i.e. is to be affirmed) iff φ is true or ψ is true (i.e. denying one commits one to affirming the other).
>
> $\varphi \vee \psi$ is false (i.e. is to be denied) iff φ is false (i.e. is to be denied) and ψ is false (i.e. is to be denied).
>
> $\varphi \to \psi$ is true (i.e. is to be affirmed) iff φ is false or ψ is true (i.e. affirming φ commits one to affirming ψ and denying ψ commits one to denying φ).

[30] A full comparison between the sort of "generalized" approach adopted here and that developed by Zach is left for future work.

[31] In Zach's generalized approach, he considers the rules as derived from specifically the conjunctive normal form, but for just these binary connective rules, it doesn't matter whether one thinks of these normal forms as conjunctive or disjunctive.

$\varphi \to \psi$ is false (i.e. is to be denied) iff φ is true (i.e. is to be affirmed) and ψ is false (i.e. is to be denied).

$\varphi \mid \psi$ is true (i.e. is to be affirmed) iff φ is false or ψ is false (i.e. affirming one commits one to denying the other).

$\varphi \mid \psi$ is false (i.e. is to be denied) iff φ is true (i.e. is to be affirmed) and ψ is true (i.e. is to be affirmed).

$\vdots$ (and so on, for $\downarrow$, $\succ\!\!-$, $\leftarrow$, and $-\!\!\prec$)

Using our schematic notation for signs that may be either positive or negative and their opposites, we can represent the pattern here as a schematic kind of De Morgan equivalence:

$$\boldsymbol{c}\langle\varphi \circ \psi\rangle \Leftrightarrow \boldsymbol{a}\langle\varphi\rangle \textit{ and } \boldsymbol{b}\langle\psi\rangle \text{ just in case } \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \Leftrightarrow \boldsymbol{a}^*\langle\varphi\rangle \textit{ or } \boldsymbol{b}^*\langle\varphi\rangle$$

And this schematic kind of De Morgan equivalence enables us to put forward schematic introduction rules for opposite stances towards logically complex sentences.

Now, as we saw, there are different ways to transpose the above equivalence schema into schematic introduction rules. One way is to have a single two-premise categorical *combining* rule corresponding to the conjunctive clause and two one-premise categorical *splitting* rules corresponding to each disjunct in the disjunctive clause. This yields the schema considered above for rules in the style of Gentzen-style's LK, with modified rules for the conditional. But the other way to do this—the way we have done here (and the way I have indicated in parentheses above)—is to represents the disjunctive clause $\boldsymbol{a}^*\langle\varphi\rangle$ *or* $\boldsymbol{b}^*\langle\psi\rangle$ in terms of the fact that that taking the opposite of one stance in the disjunctive clause (e.g. taking stance $\boldsymbol{a}$ towards φ) necessitates taking the other (e.g. $\boldsymbol{b}^*$ to ψ), and thus encode the disjunctive clause with a single *hypothetical* rule rather than two categorical splitting rules. Doing things in this way this yields the schema for the invertible Ketonen-style rules proposed here.

Now that we've explained why the schema yields rules for all of the connectives that it does, we have thereby explained why it yields rules for *just* the connectives that it does, leaving out the rules for the biconditional and exclusive disjunction. Since the truth-tables of the biconditional and the exclusive disjunction (which are each other's negation) each have two Ts and two Fs, it's not possible to put the connective and its negation in normal forms with simply a conjunction and disjunction of literals. Rather, one needs a conjunction of disjunctions or disjunction of conjunctions. Thus, to consider just the conjunctive normal forms, since the conjunctive normal forms of $p \veebar q$ and $\neg(p \veebar q)$ are $(p \vee q) \wedge (\neg p \vee \neg q)$ and $(\neg p \vee q) \wedge q \vee \neg p)$, the rules for $\veebar$ are the following (the biconditional rules are the same, but with the signs of the conclusions reversed):

$$\frac{\Gamma, -\langle\varphi\rangle \vdash +\langle\psi\rangle \quad \Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \veebar \psi\rangle} \qquad \frac{\Gamma, +\langle\varphi\rangle \vdash +\langle\psi\rangle \quad \Gamma, +\langle\psi\rangle \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\varphi \veebar \psi\rangle}$$

Thus, the premises of the positive rule for exclusive disjunction are a kind of "conjunction" of the positive rule for disjunction and the negative rule for conjunction, which are both "disjunctive" rules in the sense explicated above. Alternately, we could treat the rules for the exclusive or as a disjunction of conjunctive rules, yielding the splitting rules proposed by Francez (2014). The fact that the biconditional and exclusive disjunction require combinations of rules in this way gives concrete proof-theoretic sense to the intuition that these connectives should not be understood as primitive, but, rather, as in some way, conceptually posterior to other basic connectives such as conjunction and (inclusive) disjunction.

5 Conclusion

I have put forward a new kind of bilateral system in which the rules for all of the classical connectives are yielded by a single schema, and I have argued that these rules have a reasonable claim to being uniquely definitive of the meanings of the classical connectives. In doing so, I have shown how the bilateralist can respond both to the claim that bilateralism requires putting forward too many rules and to the claim that there are simply too many equally good options for rules on the bilatearlist proposal. However, beyond providing this response to these two potential reasons *against* bilateralism, the system I have put forth provides a new reason *for* the bilateralist set-up, given the conceptual and technical upsides of the sort of bilateral schematization I've illustrated here. Of course, there are other worries one might raise for bilateralism which I have not addressed here.[32] Still, I hope to have furthered the discussion of bilateralism in the context of proof-theoretic semantics.

6 Appendix

In this appendix, I provide proofs for the results stated in the body of the paper. The results apply to following sequent calculus, BK minus the rules for negation (which I ignore, since they're not novel to this system and it's obvious that the main result applies for them):

Axiom Schema and Structural Rule:

$$\frac{}{\Gamma, A \vdash A}\ \text{CO}$$

Where Γ and {*A*} contain only signed atomics.

$$\frac{\Gamma, A \vdash B}{\Gamma, B^{*} \vdash A^{*}}\ \text{RV}$$

Where {A} or {B} can be null.

Binary Connective Rule Schema:

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_{\circ}$$

$$\frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^{*}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}^{*}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^{*}{}_{\circ}$$

Eliminable Structural Rule:

$$\frac{\Gamma \vdash A \quad \Delta \vdash A^{*}}{\Gamma, \Delta \vdash}\ \text{BR}$$

[32]See, for instance, Dickie (2011), Restall (2020).

For the purposes of the present paper, I treat the left side of the sequents as *sets* of signed formulas. Left sides could alternately be treated as multisets or sequences, and it's worth noting that, if one does opt for such a treatment, Contraction is eliminable in this system, like Ketonen's, but I do not deal with this complication for the purposes of the present paper. Importantly, uses of RV are not taken to contribute to proof height, since it does not modify the complexity of the formulas and applications of RV can always be immediately undone through another application of RV.[33] It is possible to simplify the proofs so that one doesn't need to deal with applications of RV by working the equivalent solely left-sided system appealed to in the proof of Proposition 2, yet, since it is an important philosophical thesis of this paper that doing one's metatheory at this higher level of generality actually makes good conceptual sense in the bilateralist terms I've laid out, I have done the proof of the main theorem directly in the system proposed here.

Proposition 1: Bilateral Reductio is eliminable.

Proof: Proceeds analogously to standard Cut Elimination, but at this higher level of generality.[34] I'll refer to the schematic formula "A," in the above BR schema, which gets eliminated through an application of BR, the "BR formula" (analogously to the "Cut formula"). We induct primarily on BR formula weight with a secondary induction on BR height, where:

> **Formula Weight:** The weight of a sentence φ, $w(\varphi)$, is defined inductively where $w(p) = 1$ and $w(\varphi \circ \psi) = w(\varphi) + w(\psi) + 1$. The weight of a signed formula is simply the weight of the sentence that is signed.
>
> **BR Height:** The height of an application of BR is the sum of the heights of the proofs of the premises.

We show six ways in which a proof involving BR can be transformed to

[33]In this way, RV is treated analogously to Exchange in standard treatments of sequent calculi.

[34]See, for instance, Negri and Von Plato (2008) and Indrzejczak (2021) for analogous presentations of Cut Elimination for Ketonen-style rules.

one with either lesser BR height or BR formula weight. The schematic for the proof in which these transformations figure is as follows:

Primary Induction: On BR formula weight:

1. **Base Case:** BR on atomics is eliminable, proven by:
   (a) **Secondary Induction:** On BR height:
      i. **Base Case:** BR on atomics of height 0 is eliminable, proven by Case Zero.
      ii. **Inductive Step:** If BR on atomics of height n is eliminable, BR on atomics of height $n + 1$ is eliminable, proven by Cases One to Four.
2. **Inductive Step:** If BR on formulas of weight n is eliminable, then BR on formulas of weight $n + 1$ is eliminable, proven by Cases One through Six.

Elaborating this a bit, Case Zero shows that, when both premises of BR are axioms, so too is the Conclusion. Cases One through Four show that BR height can be reduced any case where the BR formula is not principal in both premises (a formula is said to be principal in a premise of a rule if the last rule applied was to derive that formula). Since the BR formula will never be principal in the case where it is atomic (since it won't be derived at all), this suffices to establish the inductive step of the secondary induction. For the primary inductive step, if the BR formula is not principal in both premises, then some series of transformations of type One through Four, will transform it into a proof in which the BR formula is principal in both premises, and then a transformation of type Five or Six will reduce the weight of the BR formula.

Case Zero: BR of height 0, where both premises are axioms or follow from an axiom via a single application of RV (since applications of RV don't affect proof height, such sequent are counted as axioms for our purposes here). If the left premise is an axiom, then either $A \in \Gamma$ or is some formula B such that $B \in \Gamma$ and $B^* \in \Gamma$. If the right premise is an axiom, then either $A^* \in \Delta$ or is some formula B such that $B \in \Delta$ and

$B^* \in \Delta$. So, if both premises are axioms, then either $A \in \Gamma, \Delta$ and $A^* \in \Gamma, \Delta$ or there's some formula B such that $B \in \Gamma, \Delta$ and $B^* \in \Gamma, \Delta$. Either way, $\Gamma, \Delta \vdash$ is an axiom.

Case One: BR formula is not principal in the left premise, where $\Gamma = \Gamma', \boldsymbol{c}\langle\varphi \circ \psi\rangle$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\, n}{\Gamma', A^*, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\varphi\rangle}}{\Gamma', A^* \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*_\circ}{\Gamma', \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash A}\,\text{RV} \qquad \dfrac{\vdots\, m}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\,\text{BR}$$

We have a BR of height $n + 1 + m$. We can push applications of BR up the proof tree as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\vdots\, n}{\Gamma', A^*, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\varphi\rangle}}{\Gamma', \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\varphi\rangle \vdash A}\,\text{RV} \qquad \dfrac{\vdots\, m}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}\,\text{BR}}{\Gamma', \Delta, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}\,\text{RV}}{\Gamma', \Delta \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*_\circ}{\Gamma', \Delta, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\,\text{RV}$$

to get a BR of lesser height $n + m$.

Case Two: BR formula is not principal in the left premise, where $\Gamma = \Gamma', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdots\, n}{\Gamma', A^* \vdash \boldsymbol{a}\langle\varphi\rangle} \qquad \dfrac{\vdots\, m}{\Gamma', A^* \vdash \boldsymbol{b}\langle\varphi\rangle}}{\Gamma', A^* \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_\circ}{\Gamma', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash A}\,\text{RV} \qquad \dfrac{\vdots\, k}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash}\,\text{BR}$$

We have a BR of height $max(n, m) + 1 + k$. We can push applications of BR up the proof tree as follows:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\dfrac{\vdots_n}{\Gamma', A^* \vdash \boldsymbol{a}\langle\varphi\rangle}}{\Gamma', \boldsymbol{a}^*\langle\varphi\rangle \vdash A}\,\text{RV} \quad \dfrac{\vdots_k}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{a}^*\langle\varphi\rangle \vdash}\,\text{BR}}{\Gamma', \Delta \vdash \boldsymbol{a}\langle\varphi\rangle}\,\text{RV}
\qquad
\dfrac{\dfrac{\dfrac{\dfrac{\vdots_m}{\Gamma', A^* \vdash \boldsymbol{b}\langle\varphi\rangle}}{\Gamma', \boldsymbol{b}^*\langle\varphi\rangle \vdash A}\,\text{RV} \quad \dfrac{\vdots_k}{\Delta \vdash A^*}}{\Gamma', \Delta, \boldsymbol{b}^*\langle\psi\rangle \vdash}\,\text{BR}}{\Gamma', \Delta \vdash \boldsymbol{b}\langle\psi\rangle}\,\text{RV}
}{\dfrac{\Gamma', \Delta \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma', \Delta, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash}\,\text{RV}}\,\boldsymbol{c}_\circ
$$

to get two BRs of lesser heights $n + k$ and $m + k$

Case Three: BR formula is not principal in the right premise, where $\Delta = \Delta', \boldsymbol{c}\langle\varphi \circ \psi\rangle$. Exactly analogous to Case One.

Case Four: BR formula is not principal in the right premise, where $\Delta = \Delta', \boldsymbol{c}^*\langle\varphi \circ \psi\rangle$. Exactly analogous to Case Two.

Case Five: BR formula is principal in both premises, where $A = \boldsymbol{c}\langle\varphi \circ \psi\rangle$

$$
\dfrac{
\dfrac{\dfrac{\vdots_n}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle} \quad \dfrac{\vdots_m}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}_\circ
\qquad
\dfrac{\dfrac{\vdots_k}{\Delta, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}}{\Delta \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\,\boldsymbol{c}^*{}_\circ
}{\Gamma, \Delta \vdash}\,\text{BR}
$$

We have a BR of height $max(n, m) + 1 + k + 1$. We transform the proof tree as follows:[35]

$$
\dfrac{
\dfrac{\vdots_n}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle}
\qquad
\dfrac{\dfrac{\dfrac{\vdots_m}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle} \quad \dfrac{\vdots_k}{\Delta, \boldsymbol{a}\langle\varphi\rangle \vdash \boldsymbol{b}^*\langle\psi\rangle}}{\Gamma, \Delta, \boldsymbol{a}\langle\varphi\rangle \vdash}\,\text{BR}}{\Gamma, \Delta \vdash \boldsymbol{a}^*\langle\varphi\rangle}\,\text{RV}
}{\Gamma, \Delta \vdash}\,\text{BR}
$$

Here, we have BRs of heights $m + k$ and $n + max(m, k)$, and the latter is not necessarily lesser than the original BR height, but the weight of the BR formula has decreased in both cases.

[35]Note, for simplicity's sake, we are treating contexts here as *sets* rather than multi-sets, and so there is no appeal to Contraction in the last step of this transformation.

Case Six: BR formula is principal in both premises, where $A = \boldsymbol{c}^*\langle\varphi \circ \psi\rangle$. Exactly analogous to Case Five. □

Proposition 1.1: A language L consisting in rules of this form is consistent in the sense that, if $\vdash_L A$, then $\nvdash_L A^*$.

Proof: Note first that we cannot derive the empty sequent. Since all of the axioms are of the form $\Gamma, A \vdash A$ (where A is importantly not null), and, given that the connective rules all introduce a complex formula in the conclusion sequent, the only way to derive the empty sequent would be through BR. Since BR is eliminable, the empty sequent cannot be derived. Suppose now $\vdash_L A$ and $\vdash_L A^*$. Then, by BR, we could derive the empty sequent. Since the empty sequent is not derivable, it follows in that if $\vdash_L A$ then $\nvdash_L A^*$ □

Proposition 1.2: Extending a language L to a langauge L_+ with vocabulary introduced with rules of this form constitutes a conservative extension of L in the sense that, where $\Gamma \cup \{A\}$ contains only formulas of L, if $\Gamma \nvdash_L A$, then $\Gamma \nvdash_{L_+} A$.

Proof: Follows directly from the fact that the only simplifying rule is BR, and BR is eliminable.

Proposition 1.3: The following rules—*Cut* and *Smilean Reductio (SR)*—are admissible:

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\ \text{Cut} \qquad\qquad \frac{\Gamma, A \vdash B \quad \Gamma, A \vdash B^*}{\Gamma \vdash A^*}\ \text{SR}$$

Proof: Cut and SR follow directly from BR as follows:

$$\dfrac{\dfrac{\Gamma \vdash A \quad \dfrac{\Delta, A \vdash B}{\Delta, B^* \vdash A^*}\ \text{RV}}{\Gamma, \Delta, B^* \vdash}\ \text{BR}}{\Gamma, \Delta \vdash B}\ \text{RV} \qquad\qquad \dfrac{\dfrac{\Gamma, A \vdash B \quad \Gamma, A \vdash B^*}{\Gamma, A \vdash}\ \text{BR}}{\Gamma \vdash A^*}\ \text{RV}$$

□

Proposition 2: The following fragment of BK, with just the negation rules and the rules for conjunction, disjunction, and conditional yielded by the general rule schema:

$$\frac{\Gamma \vdash -\langle\varphi\rangle}{\Gamma \vdash +\langle\neg\varphi\rangle}\ +_{\neg} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\neg\varphi\rangle}\ -_{\neg}$$

$$\frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash +\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \wedge \psi\rangle}\ +_{\wedge} \qquad \frac{\Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \wedge \psi\rangle}\ -_{\wedge}$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash +\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \vee \psi\rangle}\ +_{\vee} \qquad \frac{\Gamma \vdash -\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \vee \psi\rangle}\ -_{\vee}$$

$$\frac{\Gamma, +\langle\varphi\rangle \vdash +\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \to \psi\rangle}\ +_{\to} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash -\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \to \psi\rangle}\ -_{\to}$$

is equivalent to Ketonen's (1944) multiple conclusion sequent calculus, which I'll call "*K*":

$$\overline{\Gamma, \varphi \vdash \varphi, \Delta}$$

Where Γ, Δ, and $\{\varphi\}$ contain only atomics.

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \neg\varphi \vdash \Delta}\ \mathrm{L}_{\neg} \qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \neg\varphi, \Delta}\ \mathrm{R}_{\neg}$$

$$\frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma, \varphi \wedge \psi \vdash \Delta}\ \mathrm{L}_{\wedge} \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \varphi \wedge \psi, \Delta}\ \mathrm{R}_{\wedge}$$

$$\frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \vee \psi \vdash \Delta}\ \mathrm{L}_{\vee} \qquad \frac{\Gamma \vdash \varphi, \psi, \Delta}{\Gamma \vdash \varphi \vee \psi, \Delta}\ \mathrm{R}_{\vee}$$

$$\frac{\Gamma \vdash \varphi, \Delta \quad \Gamma, \psi \vdash \Delta}{\Gamma, \varphi \to \psi \vdash \Delta}\ \mathrm{L}_{\to} \qquad \frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma \vdash \varphi \to \psi, \Delta}\ \mathrm{R}_{\to}$$

Proof: We show first that, given Reversal, BK is equivalent to the following solely left-sided version of BK, which I call $\mathrm{BK}_{\mathrm{ls}}$:

$$\overline{\Gamma, A, A^{*} \vdash}\ ^{\mathrm{Ax}}$$

Where Γ and $\{A\}$ contain only signed atomics.

$$\frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash}\ \boldsymbol{c}_{\circ} \qquad \frac{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash \quad \Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash}\ \boldsymbol{c}^*{}_{\circ}$$

In particular, a BK_{ls} sequent of the form $\Gamma \vdash$ corresponds to an equivalence class of BK sequents of the form $\Gamma \vdash$ and $\Gamma' \vdash A^*$ for each $A \in \Gamma$, which are all immediately interprovable via Reversal, and any BK_{ls} proof corresponds to an equivalence class of BK proofs under Reversal. This equivalence between proofs is shown by induction on proof height. For the base case, any instance of the axiom schema of BK_{ls} of the form $\Gamma, A, A^* \vdash$, is obtained by Reversal from a BK axiom of the form $\Gamma, A \vdash A$. For the inductive step, we suppose that we've shown the correspondence of proofs up to height n, and we show that proofs correspond at height $n+1$ by showing that, for any application of a rule of one system, a Reversed form of the conclusion sequent can be obtained, via a rule in other system, from a Reversed form of the premise sequent(s). Insofar as Reversal does not contribute to proof height, this gives us an equivalence between proofs in the two systems.

We now provide a translation schema for going from BK_{ls} to K sequents and vice versa to show that the two systems are simply notational variants. To translate a K sequent of the form $\Gamma \vdash \Delta$ to a BK_{ls} sequent of the form $\Gamma \vdash$ let $\Gamma = \{+\langle\varphi\rangle \mid \varphi \in \Gamma\} \cup \{-\langle\varphi\rangle \mid \varphi \in \Delta\}$. Conversely, to translate a BK_{ls} sequent of the form $\Gamma \vdash$ to a K sequent of the form $\Gamma \vdash \Delta$ let $\Gamma = \{\varphi \mid +\langle\varphi\rangle \in \Gamma\}$ and $\Delta = \{\varphi \mid -\langle\varphi\rangle \in \Gamma\}$.[36] Since BK is equivalent to BK_{ls} and BK_{ls} is identical to K, BK is equivalent to K. □

Proposition 2.1: The single-connective calculus yielded by the following rules:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash -\langle\psi\rangle}{\Gamma \vdash +\langle\varphi \mid \psi\rangle}\ +_{\mid} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle \quad \Gamma \vdash +\langle\psi\rangle}{\Gamma \vdash -\langle\varphi \mid \psi\rangle}\ -_{\mid}$$

Is equivalent to the calculus proposed by Riser (1967) and Zach (2015) consisting in the following rules:

[36] As an interesting side-note, this notation precisely captures Restall's (2005) bilateral reading of multiple conclusion sequents according to which a sequent of the form $\Gamma \vdash \Delta$ expresses that the position consisting in affirming everything in Γ and denying everything in Δ is incoherent.

$$\frac{\Gamma, \varphi, \psi \vdash \Delta}{\Gamma \vdash \varphi \mid \psi, \Delta} \mid_R \qquad \frac{\Gamma \vdash \varphi, \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma, \varphi \mid \psi \vdash \Delta} \mid_L$$

Proof: Proceeds analogously to the proof of Proposition 2. Likewise for the equivalence of the Pierce's arrow rules. □

Proposition 2.2: Rules of this form are invertable, Containment not limited to atomics is admissible, and Weakening is eliminable.

Proof: Direct proofs can be straightforwardly arrived at by schematization of any of the cases in the proof of these facts for Ketonen's sequent calculus (e.g. in Negri and von Plato (2001)), but having shown the equivalence of BK and K, the fact that these facts are well-known for K suffices to establish them for BK. □

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