"Yes," "No," Neither, and Both
Bilateral Systems for the FDE Family

Ryan Simonelli

March 11, 2024

0 Introduction

Bilateral proof systems provide rules both for affirming and denying sentences. With some recent exceptions, bilateral systems have generally been proposed in the context of classical logic with the aim of providing harmonious rules for the classical connectives.[1] In classical systems, affirmation and denial are taken to be exhaustive and exclusive such that, for every sentence φ taking exactly one of these two opposite stances is correct; it can never be that neither are correct nor can it ever be that both are correct. This is formally implemented by a bilateral logic's inclusion of "coordination principles," bilateral structural rules that formally codify the relation between affirmation and denial. In this paper, I argue that, when φ is a paradoxical sentence such as the liar, it is plausibly reasonable for one to neither affirm nor deny φ, or, alternatively, for one to both affirm and deny φ. This suggests exploring bilateral logics in which the coordination principles which enforce classicality are weakened or dropped entirely. I show that, by doing this, one arrives at technically elegant and straightforwardly intuitive systems for the non-classical logics in the FDE family: K3, LP, and FDE.[2] I propose both bilateral natural deduction and bilateral

[1] Recent exceptions have been largely in the context of intuitionistic logic (e.g. [48], [3]), but see also [49]. It's worth noting, however, that, in this context "bilateralism" is not understood in terms of opposite speech acts of affirmation and denial, but, rather, in terms of the verification and falsification of sentences.

[2] K3 is Kripke's [26] "Logic of Truth" (see also Kremer [24]). LP is Priest's [30] "Logic of Paradox," first proposed by Asenjo [2]. FDE is Anderson and Belnaps's [1] logic of "First Degree Entailment." For an introductory overview of these logics, see Beall, Glanzberg and Ripley [6], Chapter 5.

sequent systems for each of these logics, and I argue that these systems constitute an advance over existing proof systems for the FDE family of non-classical logics. Existing proposed systems generally suffer from one of two significant problems: the technical problem of a lack of separability between the connective rules and negation rules or the conceptual problem of not readily admitting of an intuitive interpretation. The systems proposed here suffer from neither of these problems.

While the bilateral systems I propose are all sound and complete relative to the familiar unilateral consequence relations of the logics in the FDE family in the sense that the solely positively-signed fragment of the bilateral consequence relation generated by each system coincides with the familiar unilateral consequence relation of these logics, rather than thinking of these bilateral proof systems as simply alternative ways to generate the standard *uni*lateral consequence relations, I investigate the distinctively *bi*lateral consequence relations for the FDE family generated by these bilateral proof systems. On the definition of bilateral validity I provide, an argument with premises Γ and conclusions Δ is bilaterally valid, relative to a set of valuations V, just in case there is no $v \in V$ such that taking all of the stances in Γ is correct and taking all of the stances in Δ is incorrect, where affirming a sentence φ is correct just in case φ is (at least) true, and denying φ is correct just in case φ is (at least) false. This is a straightforward generalization of what Rumfitt [37, 224-225] [38, 808] calls "Smiley consequence," and it provides a natural definition of validity relative to which all bilateral systems for the FDE family are sound and complete given their respective sets of admissible 4-valued valuations.

I show that generalizing the logics in the FDE family from their *uni*lateral consequence relations to their *bi*lateral consequence relations has some important philosophical consequences. The most notable case, on which I focus here, is the system I call "Bilateral K3," so-called because it contains K3's unilateral consequence relation as its solely positive fragment. Notably, however, it also contains all of unilateral classical logic's consequence relation in its solely left-sided fragment. In particular, where Φ and Ψ are sets of sentences and $+\langle\Phi\rangle$ is shorthand for $\{+\langle\varphi\rangle \mid \forall\varphi \in \Phi\}$, we have $+\langle\Phi\rangle, -\langle\Psi\rangle \vDash_{B_{K3}}$ just in case $\Phi \vDash_{CL} \Psi$. On the definition of bilateral validity put forward here, $+\langle\Phi\rangle, -\langle\Psi\rangle \vDash_{B_{K3}}$ means that there is no K3 valuation such that affirming everything in Φ and denying everything in Ψ is correct. Or, in other words, $+\langle\Phi\rangle, -\langle\Psi\rangle \vDash_{B_{K3}}$ just in case affirming everything in Φ and denying everything in Ψ is, as Restall [34] says, "incoherent" or, as Ripley [35] says, "out of bounds. This solely left-sided fragment of B_{K3} turns out to just be a notational variant of the logic ST, appealed to by Ripley in response to the liar paradox, with Restall's bilateral interpretation to which Ripley appeals made explicit in the bilateral notation itself. Formulated in this bilateral setting, it becomes clear that Ripley's rejection of unilateral Cut is not a rejection of *transitivity*, but, rather, the rejection of a specific kind of bilateral *excluded middle* principle.

This result is significant in the context of the debate between "non-classical" and "substructural" approaches to semantic paradox. Ripley's approach is advertised as a substructural approach that enables us to maintain classicality. Formulated in the bilateral system put forward here, however, Ripley's approach ends up looking much less classical than it looks on Ripley's presentation; Bilateral K3 contains all unilateral classical validities on its left-hand side, but its bilateral consequence relation is clearly not classical. For instance, although BK3 validates $-\langle\varphi \vee \neg\varphi\rangle$ on the left (denying $\varphi \vee \neg\varphi$ is always incorrect), it doesn't validate $+\langle\varphi \vee \neg\varphi\rangle$ on the right (it's not the case that affirming $\varphi \vee \neg\varphi$ is always correct). So, whether or not Ripley's account is classical or not depends on whether we're talking about *unilateral* classical logic or *bilateral* classical logic. Moreover, if we're talking about the bilateral consequence relation, Ripley's account can be seen as fully structural. In particular, the structural rule of *Cut*, understood as pertaining to the *bilateral* consequence relation, can be maintained, even with the vocabulary of the a truth predicate and liar sentence added.

The paper is structured as follows. In Section 1, I lay out Rumfitt's bilateral system for classical logic. In Section 2, I motivate the *neither* or *both* responses to the liar paradox, and show how, by dropping one or both of the two coordination principles that rule out such options from Rumfitt's system, one arrives at bilateral natural deduction systems for K3, LP, and FDE. In Section 3, I adopt the generalized bilateral notation proposed by Simonelli [42] to provide a generalized formulation of these ND systems so as to provide rules for all the binary connectives (not just conjunction, disjunction, and the conditional, but also the Sheffer Stroke, Peirce's arrow, and the dual of the conditional). In Section 4, I put forward bilateral sequent systems for the FDE family, once again, in this generalized notation. In Section 5, I provide a generalized formulation of the

4-valued semantics for the FDE family and define the bilateral notion of validity relative to which all systems are sound and complete. In Section 6, I return to the two responses to the liar initially suggested in Section 2, and explicate them formally with the use of these new bilateral logics. In Section 7, I consider the implications of this approach on for Ripley's "non-transitive" solution to the liar paradox. I conclude in Section 8 by briefly responding to an "inferential expressivist" [20] objection to the account I've put forward. The Appendix provides the technical results left out of the body of the paper.

1 Rumfitt's Bilateral System for Classical Logic

A bilateral proof system of the sort proposed by Smiley [43] and Rumfitt [38] provides rules both for affirming and denying sentences. In such a system, formulas are positively or negatively signed, expressing affirmation or denial. Where φ is any sentence, $+\langle\varphi\rangle$ expresses the affirmation of φ, whereas $-\langle\varphi\rangle$ expresses the denial of φ. Rules are then provided relating signed formulas.

The most well-known bilateral system is the bilateral natural deduction system proposed by Rumfitt [38] in response to Dummett's [11] criticism of classical natural deduction having unharmonious negation rules.[3] Rumfitt shows that, by going bilateral, one is able to arrive at a perfectly harmonious system for classical logic. In Rumfitt's bilateral system, the negation rules are the following:

$$\frac{-\langle\varphi\rangle}{+\langle\neg\varphi\rangle}\ +_{\neg_I} \qquad\qquad \frac{+\langle\neg\varphi\rangle}{-\langle\varphi\rangle}\ +_{\neg_E}$$

$$\frac{+\langle\varphi\rangle}{-\langle\neg\varphi\rangle}\ -_{\neg_I} \qquad\qquad \frac{-\langle\neg\varphi\rangle}{+\langle\varphi\rangle}\ -_{\neg_E}$$

Reading the horizontal line as expressing commitment, as suggested by Incurvati and Schlöder [19], [20], the rules for affirming a negation rules say that denying φ commits one to affirming $\neg\varphi$, and affirming $\neg\varphi$ commits one to denying φ. Likewise, the rules for denying a negation say that affirming φ commits

[3] There are two systems stated in Rumfitt's article: the "more compact" system proposed by Smiley and the one Rumfitt himself proposes. The system considered here is the latter. See [27] for a discussion of these two systems and an explanation of why the latter is preferable in the context of Rumfitt's project.

one to denying $\neg\varphi$, and denying $\neg\varphi$ commits one to affirming φ. These rules are clearly harmonious, and double negation introduction and elimination are directly proven through two applications of the I-rules and E-rules respectively.

The rules for conjunction and disjunction proposed by Rumfitt are just the standard conjunction and disjunction rules from Gentzen [15], taken as the positive rules for each connective, with each connective supplemented with rules of the form of the other connective for its negative rules. Where A is any signed formula, the rules are the following :

$$\dfrac{+\langle\varphi\rangle \quad +\langle\psi\rangle}{+\langle\varphi\wedge\psi\rangle}\,+\wedge_{\mathrm{I}} \qquad \dfrac{+\langle\varphi\wedge\psi\rangle}{+\langle\varphi\rangle}\,+\wedge_{\mathrm{E}_1} \qquad \dfrac{+\langle\varphi\wedge\psi\rangle}{+\langle\psi\rangle}\,+\wedge_{\mathrm{E}_2}$$

$$\dfrac{-\langle\varphi\rangle}{-\langle\varphi\wedge\psi\rangle}\,-\wedge\mathrm{I}_1 \qquad \dfrac{-\langle\psi\rangle}{-\langle\varphi\wedge\psi\rangle}\,-\wedge\mathrm{I}_2 \qquad \dfrac{-\langle\varphi\wedge\psi\rangle \quad \begin{array}{c}\overline{-\langle\varphi\rangle}^{\,u}\\ \vdots\\ \overline{A}\end{array} \quad \begin{array}{c}\overline{-\langle\psi\rangle}^{\,v}\\ \vdots\\ \overline{A}\end{array}}{A}\,-\wedge_{\mathrm{E}}^{u,v}$$

$$\dfrac{+\langle\varphi\rangle}{+\langle\varphi\vee\psi\rangle}\,+\vee\mathrm{I}_1 \qquad \dfrac{+\langle\psi\rangle}{+\langle\varphi\vee\psi\rangle}\,+\vee\mathrm{I}_2 \qquad \dfrac{+\langle\varphi\vee\psi\rangle \quad \begin{array}{c}\overline{+\langle\varphi\rangle}^{\,u}\\ \vdots\\ \overline{A}\end{array} \quad \begin{array}{c}\overline{+\langle\psi\rangle}^{\,v}\\ \vdots\\ \overline{A}\end{array}}{A}\,+\vee_{\mathrm{E}}^{u,v}$$

$$\dfrac{-\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi\vee\psi\rangle}\,-\vee_{\mathrm{I}} \qquad \dfrac{-\langle\varphi\vee\psi\rangle}{-\langle\varphi\rangle}\,-\vee_{\mathrm{E}_1} \qquad \dfrac{-\langle\varphi\vee\psi\rangle}{-\langle\psi\rangle}\,-\vee_{\mathrm{E}_2}$$

These rules very clearly capture the duality of conjunction and disjunction. In general, in a bilateral system, where positive and negative rules have been provided for one connective, rules for its dual can be reached simply by taking the set of rules with all of the signs reversed.

Several important classically valid inferences follow from these rules. For instance, all of De Morgan's laws can be proven, given these rules, as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{+\langle\neg(\varphi\vee\psi)\rangle}{-\langle\varphi\vee\psi\rangle}\,+_{\neg_E}}{-\langle\varphi\rangle}\,-_{\vee_E}}{+\langle\neg\varphi\rangle}\,+_{\neg_I} \quad \dfrac{\dfrac{\dfrac{+\langle\neg(\varphi\vee\psi)\rangle}{-\langle\varphi\vee\psi\rangle}\,+_{\neg_E}}{-\langle\psi\rangle}\,-_{\vee_E}}{+\langle\neg\psi\rangle}\,+_{\neg_I}}{+\langle\neg\varphi\wedge\neg\psi\rangle}\,+_{\wedge_I} \qquad \dfrac{\dfrac{\dfrac{\dfrac{+\langle\neg\varphi\wedge\neg\psi\rangle}{+\langle\neg\varphi\rangle}\,+_{\wedge_E}}{-\langle\varphi\rangle}\,-_{\neg_E} \quad \dfrac{\dfrac{+\langle\neg\varphi\wedge\neg\psi\rangle}{+\langle\neg\psi\rangle}\,+_{\wedge_E}}{-\langle\psi\rangle}\,-_{\neg_E}}{-\langle\varphi\vee\psi\rangle}\,-_{\vee_I}}{+\langle\neg(\varphi\vee\psi)\rangle}\,+_{\neg_I}$$

$$
\begin{prooftree}
\AxiomC{$+\langle\neg(\varphi\wedge\psi)\rangle$}
\RightLabel{$+_{\neg_E}$}
\UnaryInfC{$-\langle\varphi\wedge\psi\rangle$}
\AxiomC{$\overline{-\langle\varphi\rangle}^{\,1}$}
\RightLabel{$+_{\neg_I}$}
\UnaryInfC{$+\langle\neg\varphi\rangle$}
\RightLabel{$+_{\vee_I}$}
\UnaryInfC{$+\langle\neg\varphi\vee\neg\psi\rangle$}
\AxiomC{$\overline{-\langle\psi\rangle}^{\,2}$}
\RightLabel{$+_{\neg_I}$}
\UnaryInfC{$+\langle\neg\psi\rangle$}
\RightLabel{$+_{\vee_I}$}
\UnaryInfC{$+\langle\neg\varphi\vee\neg\psi\rangle$}
\RightLabel{$-_{\wedge_E}{}^{1,2}$}
\TrinaryInfC{$+\langle\neg\varphi\vee\neg\psi\rangle$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{$+\langle\neg\varphi\vee\neg\psi\rangle$}
\AxiomC{$\overline{+\langle\neg\varphi\rangle}^{\,1}$}
\RightLabel{$+_{\neg_E}$}
\UnaryInfC{$-\langle\varphi\rangle$}
\RightLabel{$-_{\wedge_I}$}
\UnaryInfC{$-\langle\varphi\wedge\psi\rangle$}
\AxiomC{$\overline{+\langle\neg\psi\rangle}^{\,2}$}
\RightLabel{$+_{\neg_E}$}
\UnaryInfC{$-\langle\psi\rangle$}
\RightLabel{$-_{\wedge_I}$}
\UnaryInfC{$-\langle\varphi\wedge\psi\rangle$}
\RightLabel{$+_{\vee_E}{}^{1,2}$}
\TrinaryInfC{$-\langle\varphi\wedge\psi\rangle$}
\RightLabel{$+_{\neg_I}$}
\UnaryInfC{$+\langle\neg(\varphi\wedge\psi)\rangle$}
\end{prooftree}
$$

However, many classical validities are not provable with just these rules. For instance, we have neither $\vdash +\langle\varphi\vee\neg\varphi\rangle$ nor $+\langle\varphi\wedge\neg\varphi\rangle\vdash A$. To arrive at a classical system, bilateralists like Smiley and Rumfitt add what are called "coordination principles," bilateral structural rules which "coordinate" the opposite stances of affirmation and denial.

The key coordination principle for classical logic put forward by Smiley is what Rumfitt calls *Smilean Reductio*:[4]

$$
\begin{prooftree}
\AxiomC{B}
\AxiomC{$\overline{A}^{\,u}$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{B^*}
\RightLabel{Smiley Reduc.^{u}}
\BinaryInfC{A^*}
\end{prooftree}
$$

Here, A and B are any signed sentences, and starring a signed sentence yields the oppositely signed sentence. So, this principle says that if, given the supposition of some stance A, one can conclude two opposite stances, B and B^*, then one can conclude A^*, the opposite of A. As del Valle-Inclan [9] has noted, Smilean

[4]This is Smiley's original formulation. More recent formulations of coordination principles for classical logic (e.g.[19], [10], [9], [20]) follow Rumfitt [38, 804] in splitting up it up into the following two principles:

$$
\frac{A \quad A^*}{\perp}
\qquad\qquad\qquad
\begin{array}{c}\overline{A}^{\,u}\\ \vdots\\ \dfrac{\perp}{A^*}\;{}^{u}\end{array}
$$

Confusingly, they call the latter principle "Smiliean Reductio," when the original coinage of term by Rumfitt refers to the combined principle, which does not feature $\perp$.

Reductio is inter-derivable with the following two principles, *Bilateral Excluded Middle* and *Bilateral Explosion*:[5]

$$\dfrac{\begin{array}{c}\overline{A}^{\,u}\\ \vdots\\ B\end{array}\qquad \begin{array}{c}\overline{A^*}^{\,v}\\ \vdots\\ B\end{array}}{B}\;\text{Ex. Mid.}^{u,v} \qquad\qquad\qquad \dfrac{A \quad A^*}{B}\;\text{Explo.}$$

Splitting Smileian Reductio into Excluded Middle and Explosion in this way is perhaps a bit more perspicuous way of showing the assumptions built into Smiliean Reductio, since it's easy to see how imposing these two principles amounts to building in the assumptions of exhaustivity and exclusivity of the correctness of affirmation and denial. If we think of a proof as valid just in case it never takes us from correct stances to incorrect ones, then excluded Middle can be seen as building in the assumption that *at least one* of the opposite stances A and A^* must always be correct. After all, if B follows from A and B follows from A^*, then, given that at least one of these two opposite stances must be correct, we can conclude that B is correct. Explosion, on the other hand, can be understood as building in the assumption that *at most one* of the opposite stances A and A^* can ever be correct. After all, insofar as it can never be the case that both A and A^* are correct, inferring any stance B from these stances will never take you from correct stances to an incorrect one.

[5] The derivation of Smiliean Reductio from Excluded Middle and Explosion goes as follows:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ B\end{array}\quad B^*}{A^*}\;\text{Explo.}\qquad \overline{A^*}^{\,2}}{A^*}\;\text{Ex. Mid.}^{1,2}$$

The derivations of Excluded Middle and Explosion from Smiliean Reductio go as follows:

$$\dfrac{\dfrac{\begin{array}{c}\overline{A}^{\,1}\\ \vdots\\ B\end{array}\quad \overline{B^*}^{\,3}}{A^*}\;\text{S. Reduc.}^{1}\qquad \dfrac{\begin{array}{c}\overline{A^*}^{\,2}\\ \vdots\\ B\end{array}\quad \overline{B^*}^{\,3}}{A}\;\text{S. Reduc.}^{2}}{B}\;\text{S. Reduc.}^{3} \qquad\qquad \dfrac{A \quad A^*}{B}\;\text{S. Reduc.}^{0}$$

2 Four Bilateral ND Systems

While the assumption of the exhaustivity and exclusivity of affirmation and denial are plausible in the restricted context of classical propositional logic, we might wonder how they fare when we extend bilateral logic to contents where classical logic has been called into question. Perhaps the most famous context is in debates surrounding the following sentence:

> $\boldsymbol{\lambda}$**:** λ is not true.

Is λ true? Suppose we say "Yes," affirming that λ is true. If it's true, then what it says is true, but what it says is that it is not true, and so, if that's true, then it's not true. It seems, then, that if we say "Yes," affirming that λ is true, we're committed to saying "No," denying that λ is true. Suppose, then, that we say "No," denying that λ is true. If it's not true, then, given that what it says is that it's not true, it says something true, and so it's true. So, if we say "Yes," in response to the question of whether λ is true, we're committed to saying "No," and if we say "No," we're committed to saying "Yes." Thus, if we say either "Yes" or "No," then, we're committed to saying both "Yes" and "No." What, then, should we say in response to the question of whether λ is true? There are, I think, two plausible responses.

The first response is say nothing. That is, in response to the question of whether λ is true, we say neither "Yes" nor "No." This seems like what we ought to do in response to the question insofar as we don't want to commit ourselves to saying both "Yes" and "No." As we've just seen, if we say either, we commit ourselves to saying both. So, we might say neither.[6] If we take this line in response to the liar, then we should maintain that affirming the liar is incorrect, but that this doesn't mean that denying the liar is correct. On the contrary, denying the liar is incorrect too.

The second response to say something. Now, as we've seen, insofar as we

[6]Crucially, saying neither "Yes" nor "No" is distinct from saying "Neither 'Yes' nor 'No'." Saying that latter thing is tantamount to affirming $\neg(\lambda \vee \neg\lambda)$, and, doing that (as we'll see officially shortly) will commit one to both affirming λ and denying φ, precisely the thing one wants avoid in being silent in response to the question of whether λ. So, if one wishes to be silent in response to the question of λ, one should also be silent in response to the question of $\lambda \vee \neg\lambda$, as taking either positive or negative stance in response to this question will commit one to taking both in response to the question of λ.

say "Yes" in response to the question of whether λ is true, we are committed to saying "No" to this question, and insofar as we say "No" to this question, we are committed to saying "Yes." Accordingly, insofar as we say something, the only thing we can coherently say is both "Yes" and "No." If we take this line in response to the liar, then we should maintain that affirming the liar is correct, and denying it is also correct, but that doesn't mean, for instance, that affirming that the moon is made of cheese is correct. On the contrary, it's incorrect to affirm that the moon is made of cheese.[7]

I do not know which of these two responses is to be preferred, but they both seem *prima facie* plausible to me. The proponent of the first response denies that affirmation and denial are *exhaustive*: there are some sentences φ (for instance, λ) such that neither affirming φ nor denying φ is correct. Accordingly, one cannot assume that one must be correct, as one implicitly does in using Bilateral Excluded Middle. The proponent of the second response denies that affirmation and denial are *exclusive*: there are some sentences φ (for instance, λ) such that both affirming φ and denying φ are correct. Accordingly, one cannot assume that they can never both be correct, as one implicitly does in using Explosion. Given that rejecting either of these coordination principles can be motivated on these grounds, it is natural to wonder what consequence relations one gets of excludes one or both of these principles from a bilateral system. It turns out, perhaps unsurprisingly, that we get the bilateral versions of some familiar logics. In particular, we get the following four:

1. $\mathbf{BN_{CL}}$**:** Operational rules + Explosion + Excluded Middle
2. $\mathbf{BN_{K3}}$**:** Operational rules + Explosion
3. $\mathbf{BN_{LP}}$**:** Operational rules + Excluded Middle
4. $\mathbf{BN_{FDE}}$**:** Operational rules

[7]It's worth note that this line, while recognizably dialetheic, diverges from Priest's [32, 103-106] proposal for understanding the relation between denial and negation (see also Smiley and Priest [44]). Notably, Priest attempts to *sever* the tight inferential connection between denial and negation maintained by bilateralists, maintaining that one should assert "It's not the case that λ" but that one should not deny that λ. Insofar as the bilateralist account of negation is an attractive, I think there is good reason to try avoid Priest's conclusion if we can. This paper shows how we can.

Proposition 1: *Soundness and completeness with respect to unilateral semantics:* BN_{FDE} proves $+\langle\varphi_1\rangle, +\langle\varphi_2\rangle \ldots + \langle\varphi_n\rangle \vdash +\langle\psi\rangle$ just in case $\varphi_1, \varphi_2 \ldots \varphi_n \vDash_{FDE} \psi$. Likewise for BN_{LP} and BN_{K3}

Proof: We've already seen that BN_{CL} is just Rumfitt's classical bilateral system, modulo the conditional rules. To see that these other bilateral natural deduction systems yield the (unilateral) consequence relations of these other logics, we note that Priest [33] has already provided unilateral natural deduction systems for FDE, LP, and K3, that are sound and complete with respect to their respective semantics.[8] The FDE rules are the (positive) conjunction and disjunction rules, double negation introduction and elimination, and all four directions of De Morgan. Since this system contains the conjunction and disjunction rule, double negation is immediately derivable from the negation rules, and we've shown that all four directions of De Morgan are derivable, this system is complete for FDE. For LP, Priest ads $\vdash \varphi \vee \neg\varphi$ as a primitive rule, and, for K3, $\varphi \wedge \neg\varphi \vdash \psi$ is added. Note, then, that with the coordination principle of Excluded Middle, one can derive the Law of Excluded Middle as Follows:

$$\dfrac{\dfrac{\dfrac{}{+\langle\varphi\rangle}{}^{1}}{+\langle\varphi \vee \neg\varphi\rangle}{+_{\vee_I}} \qquad \dfrac{\dfrac{\dfrac{}{-\langle\varphi\rangle}{}^{2}}{+\langle\neg\varphi\rangle}{+_{\neg_I}}}{+\langle\varphi \vee \neg\varphi\rangle}{+_{\vee_I}}}{+\langle\varphi \vee \neg\varphi\rangle}\,\text{Ex. Mid.}^{1,2}$$

And, with Explosion, one can derive $+\langle\psi\rangle$ from a contradiction $+\langle\varphi \wedge \neg\varphi\rangle$ as follows:

$$\dfrac{\dfrac{+\langle\varphi \wedge \neg\varphi\rangle}{+\langle\varphi\rangle}{+_{\wedge_E}} \qquad \dfrac{\dfrac{+\langle\varphi \wedge \neg\varphi\rangle}{+\langle\neg\varphi\rangle}{+_{\wedge_E}}}{-\langle\varphi\rangle}{+_{\neg_E}}}{+\langle\psi\rangle}\,\text{Explo.}$$

Since all of Priest's rules are derivable in these systems, all systems are complete. For soundness, it's sufficient to note that versions of all of the negative rules in this system in which the minus signs have been replaced with negations are derivable in Priest's system, and the negative rules are derivable in this system

[8] See also Field [13, 79-82]

from those. Likewise, given $\vdash \varphi \vee \neg\varphi$ and $\varphi \wedge \neg\varphi \vdash \psi$, the coordination principles of Explosion and Excluded middle, with negations rather than minus signs, can be derived in Priest's system given the operational rules. $\square$

2.1 Advantage Over Existing Natural Deduction Systems

Though these bilateral natural deduction systems are obviously quite close to Priest's, what's crucial is that all of these rules are *separable*, in that the rules for each connective only contain that connective. Appreciating the separability point is crucial here, and it will be crucial in what is to follow, so it's worth taking a moment to emphasize it. It is very important that the minus sign, "$-$," in a bilateral system is not a funny kind of negation. Unlike the logical connective, $\neg$, which, prefixed to a sentence, yields another sentence that figures in the recursive formation rules (and thus, given that φ is a sentence, $\neg\varphi$ is a sentence, $\psi \vee \neg\varphi$ is a sentence, $\neg(\psi \vee \neg\varphi)$ is a sentence, and so on), $-$, is not a logical connective at all. A more perspicuous notation, which would make the difference between negation and the minus sign very clear would be to not use signs at all, but, rather to *color* sentences, for instance, green or red to express their affirmation or denial. In such a notation, there *are no signs*, just sentences, which may be colored in one of two ways. This makes it clear that the positive rules for conjunction and disjunction in this bilateral system *just are* the standard natural deduction rules from Gentzen—no extra symbols are added to these rules. It's just that, Gentzen's rules codify only one aspect of the inferential role of conjunctions and disjunction: the role insofar as they are asserted, rather then denied. Of course, using signs is much more convenient, and I will shortly go on to introduce an even more convenient notation that schematizes over signs. Officially, however, it is perhaps best to think of all of this notation as a convenient way of talking about systems in which sentences are colored rather than signed.

Perhaps the most significant advance of this natural family of deduction systems for the FDE family over existing natural deduction systems is in the rules for negation. In standard natural deduction systems for the FDE family of the sort proposed by Priest, there are no rules for introducing or eliminating a single negation. Rather, there are only *double* negation rules and rules for making inferences to and from negated conjunctions and disjunctions. That is, the rules of Priest's system are the following:

$$\frac{\varphi}{\neg\neg\varphi}\ \neg_I \qquad\qquad \frac{\neg\neg\varphi}{\varphi}\ \neg_E$$

These rules by themselves simply don't tell us that $\neg$ means *not*. This should be obvious, but, to see this, simply note that the connective "$\neg$" could mean *it's true that*, and we could have just the same rules. Surely, from φ one can infer "It's true that it's true that φ," and from "It's true that it's true that φ" we can infer φ. What this means is that these rules by themselves don't suffice to specify the meaning of the negation connective that figures in these natural deduction systems. The meaning of negation must be given in part by its interaction with the other connectives. By contrast, the rules for negation presented here are the very same rules given in bilateral natural deduction systems for classical logic. They directly specify the conditions and consequences of affirming or denying a negated sentence: one is to affirm a negation just in case one is to deny the negatum and one is to deny a negation just in case one is to affirm the negatum. Given that there are the same negation rules across all of these logics one can maintain that the meaning of negation is codified by these rules and that negation has the same meaning whether one is reasoning in CL, K3, LP or FDE. All that changes is the relation between affirmation and denial: whether these opposite stances are taken to be exclusive and exhaustive, just exclusive, just exhaustive, or neither.

3 Generalized Sequent Formulation

I've presented bilateral natural deduction systems for the FDE family. The rules for negation, conjunction, and disjunction are just those proposed by Rumiftt (2000), with different coordination principles determining the different logics. It is easy to see that we can give rules for the conditional of exactly the same form. In fact, we can give rules for *all* of the binary connectives—not just the conditional, but also the Sheffer Stroke, Peirce's arrow, and the conditional's dual—of exactly the same form. To do this, I'll follow Simonelli [42] in adopting a generalized bilateral notation to make the statement of the whole system more concise.[9] Rather than using + or − to state the rules, I'll use variables such as $\boldsymbol{a}$ and $\boldsymbol{b}$ to indicate signs that may be either + or − along with a function * that maps + to − and − to +. So, for any signed formula of the form $\boldsymbol{a}\langle\varphi\rangle$, where $\boldsymbol{a} \in \{+,-\}$, if $\boldsymbol{a} = +$ then $\boldsymbol{a}^* = -$, and if $\boldsymbol{a} = -$ then $\boldsymbol{a}^* = +$. Stating the natural deduction system in sequent notation (to facilitate easy comparison with the sequent systems to come), the natural deduction system for the FDE family is the following:

Structural Rules:

$$\frac{}{A \vdash A}\ \text{Reflexivity} \qquad \frac{\Gamma \vdash A}{\Gamma', \Gamma \vdash A}\ \text{Weakening} \qquad \frac{\Gamma, A \vdash B \quad \Gamma \vdash A}{\Gamma \vdash B}\ \text{Cut}$$

Connective Rules:

$$\frac{\Gamma \vdash -\langle\varphi\rangle}{\Gamma \vdash +\langle\neg\varphi\rangle}\ +_{\neg_I} \qquad \frac{\Gamma \vdash +\langle\neg\varphi\rangle}{\Gamma \vdash -\langle\varphi\rangle}\ +_{\neg_E}$$

$$\frac{\Gamma \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\neg\varphi\rangle}\ -_{\neg_I} \qquad \frac{\Gamma \vdash -\langle\neg\varphi\rangle}{\Gamma \vdash +\langle\varphi\rangle}\ -_{\neg_E}$$

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_{\circ\mathrm{I}} \qquad \frac{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_1} \qquad \frac{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_2}$$

$$\frac{\Gamma \vdash \boldsymbol{a}^*\langle\varphi\rangle}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*_{\circ\mathrm{I}_1} \qquad \frac{\Gamma \vdash \boldsymbol{b}^*\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*_{\circ\mathrm{I}_2}$$

$$\frac{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \quad \Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash A \quad \Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash A}{\Gamma \vdash A}\ \boldsymbol{c}^*_{\circ\mathrm{E}}$$

Coordination Principles:

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A^*}{\Gamma \vdash B}\ \text{Explo.} \qquad \frac{\Gamma, A \vdash B \quad \Gamma, A^* \vdash B}{\Gamma \vdash B}\ \text{Ex. Mid.}$$

[9] This notation is similar to Smullyan's [45, 21-23] "unifying notation," but both more flexible and more conceptually transparent.

Here, the binary connective rules have been stated in terms of a *schema* such that, for any assignment of signs to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$,the rules for some binary connective are given. In particular, the rules for the following connectives are given by the following assignments (where | is the Sheffer Stroke, $\downarrow$ is Peirce's arrow, and $\succ\!\!-$ is the dual of the conditional):[10]

$$\begin{array}{ll} \wedge\text{: } \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = + & \vee\text{: } \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = - \\ |\text{: } \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = - & \downarrow\text{: } \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = + \\ \rightarrow\text{: } \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = - & \succ\!\!-\text{: } \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = + \\ -\!\!\prec\text{: } \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = + & \leftarrow\text{:} \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = - \end{array}$$

So, the natural deduction systems for the members of the FDE family provided here technically contain all of these rules. However, one can take whichever fragment one likes. Thus, for instance, if one wants a separable, harmonious, and truth-functionally complete ND system for LP with as few connectives as possible, one can take just the rules for the Sheffer Stroke along with the the coordination principle of Excluded Middle.

4 Four Sequent Systems

I'll now introduce a family of multiple conclusion sequent systems using the same generalized notation:

Structural Rules:

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{Reflex.} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ \text{Weak.} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ \text{Cut}$$

Connective Rules:

$$\frac{\Gamma, -\langle\varphi\rangle \vdash \Delta}{\Gamma, +\langle\neg\varphi\rangle \vdash \Delta}\ +_{\neg_L} \qquad \frac{\Gamma, +\langle\varphi\rangle \vdash \Delta}{\Gamma, -\langle\neg\varphi\rangle \vdash \Delta}\ -_{\neg_L}$$

[10] See Simonelli [42] for an explication of the use of this symbol for the dual of the conditional and the connection with the use of the same symbol by Ayhan [3] and Wansing [48]. This symbol has the advantage over $\twoheadleftarrow$, used by Zach [52], in that it is a primitive symbol. However, it does have the disadvantage of being the same symbol used by Peirce [29] for "inclusion"

$$\frac{\Gamma \vdash -\langle\varphi\rangle, \Delta}{\Gamma \vdash +\langle\neg\varphi\rangle, \Delta}\ +_{\neg R} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle, \Delta}{\Gamma \vdash -\langle\neg\varphi\rangle, \Delta}\ -_{\neg R}$$

$$\frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{L}}} \qquad \frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle, \Delta \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle, \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{R}}}$$

$$\frac{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash \Delta \quad \Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{L}}} \qquad \frac{\Gamma \vdash \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{b}^*\langle\psi\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle, \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{R}}}$$

Coordination Principles:

$$\frac{}{\Gamma, A, A^* \vdash \Delta}\ \text{Explo.} \qquad \frac{}{\Gamma \vdash A, A^*, \Delta}\ \text{Ex. Mid.}$$

Like Ketonen's [23] classical sequent calculus, proof of a sequent is constructed by root-first proof search, and this yields a decision procedure for proving or refuting sequents. Notably, both Cut and Weakening are eliminable, and Reflxeivity can be restricted to atomics.[11] For our purposes, however, it will be useful to keep the structural rules in there for now.

The coordination principles of Explosion and Excluded middle are now given as possible axiom schemas one might have in addition to Reflexivity. So, whereas, in the ND system, the coordination principles of Explosion and Excluded Middle were presented at the *meta-inferential* level, relating sequents, here they are presented at the (first-order) *inferential* level.

Considering all of the possibilities for coordination principles, we'll define the following four sequent systems:

1. $\mathbf{BS_{CL}}$**:** Operational rules + Explosion and Excluded Middle
2. $\mathbf{BS_{LP}}$**:** Operational rules + Excluded Middle
3. $\mathbf{BS_{K3}}$**:** Operational rules + Explosion
4. $\mathbf{BS_{FDE}}$**:** Operational rules

Proposition 2: Each sequent system derives all of the rules of the corresponding natural deduction system.

[11] The proofs are given in the Appendix.

Proof: Considering first the operational rules, all of the operational rules of the ND system presented above are derivable in this system. The introduction rules of the ND system are directly derivable from the right rules of this sequent system (through a single application of Weakening in the case of the $\boldsymbol{c}^*{}_\circ$ rules). The derivability of the elimination rules follows from the invertibility of the rules:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}\langle\varphi\rangle}\,\text{Reflex.}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \boldsymbol{a}\langle\varphi\rangle}\,\boldsymbol{c}^*{}_\circ \qquad \Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle}\,\text{Cut} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{b}\langle\psi\rangle}\,\text{Reflex.}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \boldsymbol{b}\langle\psi\rangle}\,\boldsymbol{c}^*{}_\circ \qquad \Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}\,\text{Cut}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \qquad \dfrac{\dfrac{}{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{b}^*\langle\psi\rangle}\,\text{Reflex.} \qquad \dfrac{}{\Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{b}^*\langle\psi\rangle}\,\text{Reflex.}}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{b}^*\langle\psi\rangle}\,\boldsymbol{c}^*{}_\circ}{\Gamma \vdash \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{b}^*\langle\psi\rangle}\,\text{Cut} \qquad \Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash A}{\Gamma \vdash \boldsymbol{b}^*\langle\psi\rangle, A}\,\text{Cut} \qquad \Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash A}{\Gamma \vdash A}\,\text{Cut}$$

Once again, non-atomic Reflexivity, Weakening, and Cut are all eliminable in this system, and so the ND rules are all admissible in the stripped down system that does not have these rules.

Consider now the coordination principles. Given inferential Excluded Middle and Explosion, metainferential Excluded Middle and Explosion can be derived with Cut as follows:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A, A^*, B}\,\text{Ex. Mid.} \qquad \Gamma, A^* \vdash B}{\Gamma \vdash A, B}\,\text{Cut} \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\,\text{Cut} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, A, A^* \vdash B}\,\text{Explo.} \qquad \Gamma \vdash A}{\Gamma, A^* \vdash B}\,\text{Cut} \qquad \Gamma \vdash A^*}{\Gamma \vdash B}\,\text{Cut}$$

Thus, all four sequent systems derive all rules of their corresponding natural deduction system. □

Another pair of coordination principles are worth considering are two that I call "In" and "Out" respectively:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, A^* \vdash \Delta}\ \text{In} \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^*}\ \text{Out}$$

Proposition 3: Explosion and In are inter-derivable and Excluded Middle and Out are inter-derivable.

Proof: It is easy to see that Excluded Middle is immediately derivable from Reflexivity with Out and Explosion is immediately derivable from Reflexivity with In. Likewise, given Cut, Out is immediately derivable from Excluded Middle and In is immediately derivable from Explosion:

$$\frac{\Gamma, A \vdash \Delta \quad \overline{\Gamma \vdash A, A^*, \Delta}\ \text{Ex. Mid.}}{\Gamma \vdash A^*, \Delta}\ \text{Cut} \qquad\qquad \frac{\Gamma \vdash A, \Delta \quad \overline{\Gamma, A, A^* \vdash \Delta}\ \text{Explo.}}{\Gamma, A^* \vdash \Delta}\ \text{Cut}$$

□

Thus, rather than adding Excluded Middle or Explosion as an additional axiom schema, another way to get a logic with the consequence relation of Bilateral LP or Bilateral K3 is with the addition of coordination principles is to add Out or In respectively.[12] However, there are proof-theoretic benefits of our treatment of coordination principles as axiom schemas, as it easily facilitates root-first proof search, making the completeness proofs straightforward.

In and Out together yield (the multiple conclusion version of) the principle that Smiley (1996) dubs Reversal:

$$\frac{\Gamma, A \vdash B, \Delta}{\Gamma, B^* \vdash A^*, \Delta}\ \text{Reversal}$$

Where {A} or {B} can be null.

It's obvious that, with this rule, having both left rules and right rules is redundant, since one can put forward calculus containing only rules for deriving formulas on one side of the turnstile and be able to derive any formula on the other side the turnstile by deriving its opposite and using Reversal. Thus, given that Reversal is derivable in BS_{CL}, half of the rules are redundant. In the bilateral

[12]This is the route that Shapiro [40] takes in his proposal of a 4-sided sequent calculus for the FDE family where In and Out show up as what he calls "shift" rules. See note below for the correspondence.

sequent calculus for classical logic proposed by Simonelli [42], Reversal is the only coordination principle, and there are only right rules: one positive rule and one negative rule for each connective. For the non-classical members of the FDE family, however, there is a crucial asymmetry between the two sides of the turnstile, and so having both left and right rules is necessary.

4.1 Advantage Over Existing Sequent Systems

These bilateral sequent calculi provide the same advantages over standard multiple conclusion sequent calculi for the FDE family (e.g. Beall [4]) that the natural deduction systems proposed above provide over standard natural deduction systems. In particular, all of the connective rules are separable from the negation rules and there are proper rules for negation. This is especially notable insofar as the lack of separability and proper negation rules in the standard unilateral sequent calculi the FDE family leads Beall [5], a proponent of FDE, to conclude that "logic itself" says nothing about negation other than its "interaction with other logical connectives (e.g., $\neg\wedge$, $\neg\vee$ etc.)," (15). Since Beall takes it that, "on the FDE picture of logic there simply is no stand-alone negation behavior that logic itself describes" (16), Beall comes to the radical conclusion that "there is no logical negation" at all. The sequent systems for the FDE family proposed here enable one to avoid this conclusion. In all logics in the FDE family—even FDE where there are no coordination principles—negation is still a kind of "flip-flop" operator, as it is on a standard sequent calculus for classical logic. Once again, if denying φ is correct, then affirming $\neg\varphi$ is correct and vice versa, and, if affirming φ is correct, then. This contrasts a negation operator with a truth-operator which precisely *doesn't* flip-flop between affirmation and denial.

Other sequent systems for the FDE family that aim to restore the negation's as "flip-flop" operator have been proposed, such as Wintein's [51] 4-signed sequent system, Fjellstad's [14] dual 2-sided sequent system, and Shapiro's [40] 4-sided sequent system. Technically, all of these systems are very close to one another and with the sequent systems proposed here. In fact, with some minor variations, there is a translation procedure for going between all three of these non-standard systems, and for going between any one of them and the bilateral system proposed here.[13] However, there are significant conceptual reasons to prefer the sequent system proposed here over these other non-standard sequent systems which are technically very close to it. This proof system admits of a straightforwardly intuitive interpretation, whereas the same cannot be said for these other non-standard systems. Consider, for instance, the rules for negation in Shapiro's 4-sided sequent system:[14]

$$\frac{\Gamma; \Sigma \vdash \Delta; \varphi, \Theta}{\Gamma, \neg\varphi; \Sigma \vdash \Delta; \Theta}\ L_{\neg_1} \qquad \frac{\Gamma; \Sigma \vdash \varphi, \Delta; \Theta}{\Gamma; \Sigma, \neg\varphi \vdash \Delta; \Theta}\ L_{\neg_2}$$

$$\frac{\Gamma; \Sigma, \varphi \vdash \Delta; \Theta}{\Gamma; \Sigma \vdash \neg\varphi, \Delta; \Theta}\ R_{\neg_1} \qquad \frac{\Gamma, \varphi; \Sigma \vdash \Delta; \Theta}{\Gamma; \Sigma \vdash \Delta; \neg\varphi, \Theta}\ R_{\neg_2}$$

These rules obviously restore negation's status as a "flip-flop" operator. However, it's simply not clear what it is actually flipping and flopping between. Wintien's [51, 539-540] suggestion for how to interpret these 4-sided sequents comes from Restall's [34] bilateral interpretation of 2-sided sequents.[15] On Restall's account, a sequent of the form $\Gamma \vdash \Delta$ is read as saying that affirming everything in Γ and denying everything. Wintein suggests that, by invoking a distinction between *strict* and *tolerant* assertion (cf. Ripley [35, 153-155]), we can get a "quadrilateral" interpretation of 4-sided sequnents.[16] Thus, a four-sided sequent of the form $\Gamma; \Sigma \vdash \Delta; \Theta$, can be read as saying that strictly asserting everything in Γ, tolerantly asserting everything in Σ, tolerantly denying everything in Δ, and strictly denying everything in Θ is incoherent or "out of bounds." I have no decisive objection to the "quadrilateral" approach. However, it seems clearly desirable to work with a *univocal* notion of affirmation and denial if possible. On the bilateral systems proposed here, there is a single, univocal notion of affirmation and denial, a single, univocal notion of correctness, and a single univicol notion of validity pertaining to all of these systems; all that differs are

[13] See Shapiro [40] for translations between the three non-standard systems. There is a direct one-to-one translation between Shapiro and Wintein's system and the one proposed here. To translate a 4-sided sequent of the form $\Phi; X \vdash Z; \Psi$ to a 2-sided bilateral sequent of the form $\Gamma \vdash \Delta$, let $\Gamma = \{+\langle\varphi\rangle \mid \varphi \in \Phi\} \cup \{-\langle\psi\rangle \mid \psi \in \Psi\}$ and $\Delta = \{+\langle\chi\rangle \mid \chi \in X\} \cup \{-\langle\zeta\rangle \mid \zeta \in Z\}$.

[14] I'll focus here on Shapiro's 4-sided sequent system, but the same basic remarks go for all of these systems.

[15] Shapiro has also suggested this interpretation (personal communication).

[16] See also Wintein [50] for a further development of the strict/tolerant distinction applied in this interpretation.

the coordination principles relating affirmation and denial. Let us now turn to the semantics to make these ideas of correctness precise.

5 Generalized Semantics and Bilateral Validity

The basic idea of the semantic approach is simply that affirming some sentence is correct just in case that sentence is true and denying a sentence is correct just in case that sentence is false. To spell out this idea formally, I'll adopt a set of notational conventions from Simonelli [42]. First, we'll define first the following function:

> **Correctness Function:** The *correctness function* $[]$ is a function from $\{+, -\}$ to $\{1, 0\}$ mapping $+$ to 1 and $-$ to 0.

In the context of classical logic, where each sentence is assigned 1 or 0 and not both, we can then define correctness as follows:

> **Classical Correctness:** Taking some stance $\boldsymbol{a}$ towards some sentence φ, $\boldsymbol{a}\langle\varphi\rangle$, is *correct*, relative to some valuation v, just in case $v(\varphi) = [\boldsymbol{a}]$.

Simonelli proposes reading the expression $[\boldsymbol{a}]$ as "the truth value that would make stance $\boldsymbol{a}$ correct." Thus, given that $v(\varphi) = [+]$ just in case $v(\varphi) = 1$ and $v(\varphi) = [-]$ just in case $v(\varphi) = 0$, we can say that φ's truth value is the one that would make affirmation correct just in case φ is true, and φ's truth value is the one that would make denial correct just in case φ is false.

Using these notational conventions and the assignment of signs to the variables $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ presented above, it is possible to state the semantics for all of the classical connectives in general terms. Let a *classical valuation* v be any function from $\mathcal{L} \to \{1, 0\}$ that assigns an element of $\{1, 0\}$ to each atomic sentence and recursively assigns values to complex sentences as follows:

$$v(\neg\varphi) = \begin{cases} 1, & \text{iff } v(\varphi) = 0 \\ 0, & \text{iff } v(\varphi) = 1 \end{cases}$$

$$v(\varphi \circ \psi) = \begin{cases} [\boldsymbol{c}], & \text{iff } v(\varphi) = [\boldsymbol{a}] \text{ and } v(\psi) = [\boldsymbol{b}] \\ [\boldsymbol{c}^*], & \text{iff } v(\varphi) = [\boldsymbol{a}^*] \text{ or } v(\psi) = [\boldsymbol{b}^*] \end{cases}$$

Recall our assignment of signs to sign variables:

$\wedge$: $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = +$	$\vee$: $\boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = -$
$\mid$: $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = -$	$\downarrow$: $\boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = +$
$\rightarrow$: $\boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = -$	$\succ\!\!-$: $\boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = +$
$-\!\!<$: $\boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = +$	$\leftarrow$:$\boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = -$

Given these assignments, the second clause in the definition of a classical valuation specifies the truth-conditions of all of the classical connectives at once. Thus, for instance, we have the following instance for conjunction where $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = +$:

$$v(\varphi \wedge \psi) = \begin{cases} 1, & \text{iff } v(\varphi) = 1 \text{ and } v(\psi) = 1 \\ 0, & \text{iff } v(\varphi) = 0 \text{ or } v(\psi) = 0 \end{cases}$$

Likewise for all other connectives. Given this general specification of semantic clauses and a proof system specified in generalized bilateral notation, one can prove soundness and completeness at this level of generality.

While Simonelli presents this generalized approach only for the semantics of classical logic, it is straightforwardly extended to four-valued semantics with values $\varnothing, \{1\}, \{0\}, \{1, 0\}$.[17] All of the same clauses hold, but we simply swap the identity sign with that of containment. Thus, we can first define the following notion of correctness, applicable to four-valued logics:

> **4-valued Correctness:** Taking some stance $\boldsymbol{a}$ towards some sentence φ, $\boldsymbol{a}\langle\varphi\rangle$, is *correct*, relative to some valuation v, just in case $[\boldsymbol{a}] \in v(\varphi)$.

Now, rather than reading the expression $[\boldsymbol{a}]$ as "*the* truth value that would make stance $\boldsymbol{a}$ correct," we read it "*a* truth value that would make stance $\boldsymbol{a}$ correct," where it's possible that the valuation of a sentence contains no such truth value or that it contains more than one such truth value. So, in words, this definition of correctness says that taking some stance, be it affirmation or denial, towards some sentence is correct just in case the value of that sentence contains some truth value that would make that stance correct. So, for instance, if φ is just true, then affirming φ is correct and denying ψ is incorrect. If φ is both true and false,

[17] See Dunn [12] and Priest [31] for this presentation of the semantics.

then affirming φ is correct, and so is denying φ. If φ is neither true nor false, then neither affirming φ nor denying φ is correct; that is, both stances are incorrect. Note that correctness, as it's defined here, is a binary property. For any stance $\boldsymbol{a}$ towards any sentence φ on any valuation v, $\boldsymbol{a}\langle\varphi\rangle$ is either correct or incorrect and not both.

Now, let a *four-valued valuation* v be any function from $\mathcal{L} \to \{\varnothing, \{1\}, \{0\}, \{1,0\}\}$ that assigns an element of this set to each atomic sentence p and recursively assigns values to complex sentences as follows:

$$v(\neg\varphi) \ni \begin{cases} 1, & \text{iff } 0 \in v(\varphi) \\ 0, & \text{iff } 1 \in v(\varphi) \end{cases}$$

$$v(\varphi \circ \psi) \ni \begin{cases} [\boldsymbol{c}], & \text{iff } [\boldsymbol{a}] \in v(\varphi) \text{ and } [\boldsymbol{b}] \in v(\psi) \\ [\boldsymbol{c}^*], & \text{iff } [\boldsymbol{a}^*] \in v(\varphi) \text{ or } [\boldsymbol{b}^*] \in v(\psi) \end{cases}$$

Thus, for instance, we have the following instance for conjunction whose assignment is $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = +$:

$$v(\varphi \wedge \psi) \ni \begin{cases} 1, & \text{iff } 1 \in v(\varphi) \text{ and } 1 \in v(\psi) \\ 0, & \text{iff } 0 \in v(\varphi) \text{ or } 0 \in v(\psi) \end{cases}$$

And it is easy to confirm that this gives us the 4-valued truth-table for conjunction:

$\wedge$	$\{1\}$	$\{1,0\}$	$\varnothing$	$\{0\}$
$\{1\}$	$\{1\}$	$\{1,0\}$	$\varnothing$	$\{0\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{0\}$	$\{0\}$
$\varnothing$	$\varnothing$	$\{0\}$	$\varnothing$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$

Likewise for all of the other connectives. For instance, we get the following valuation function for the Sheffer Stroke who's assignment is $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = -$:

$$v(\varphi \mid \psi) \ni \begin{cases} 0, & \text{iff } 1 \in v(\varphi) \text{ and } 1 \in v(\psi) \\ 1, & \text{iff } 0 \in v(\varphi) \text{ or } 0 \in v(\psi) \end{cases}$$

$\mid$	$\{1\}$	$\{1,0\}$	$\varnothing$	$\{0\}$
$\{1\}$	$\{0\}$	$\{1,0\}$	$\varnothing$	$\{1\}$
$\{1,0\}$	$\{1,0\}$	$\{1,0\}$	$\{1\}$	$\{1\}$
$\varnothing$	$\varnothing$	$\{1\}$	$\varnothing$	$\{1\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

And it is easy to see that the $\varphi \mid \psi$ is equivalent to $\neg(\varphi \wedge \psi)$, as it should be. In the same way, this single generic specification of truth-conditions yields the 4-valued truth-tables of disjunction, Peirce's arrow, the conditional (in both directions), and the dual of the conditional (in both directions): all eight of our connectives given by the assignment of signs shown above.

Having defined all 4-valued valuations, we can now define the set of admissible valuations for each of the logics in the FDE family:

> **Admissible Valuations:** The admissible valuations for CL, LP, K3, and FDE are the subsets of the above set of valuations where atomics are mapped only to certain designated truth-values:
>
> 1. **CL:** All valuations $\mathcal{A} \to \{\{1\},\{0\}\}$
> 2. **LP:** All valuations $\mathcal{A} \to \{\{1\},\{0\},\{1,0\}\}$
> 3. **K3:** All valuations $\mathcal{A} \to \{\varnothing,\{1\},\{0\}\}$
> 4. **FDE:** All valuations $\mathcal{A} \to \{\varnothing,\{1\},\{0\},\{1,0\}\}$

We can now define the following generic notion of bilateral validity:

> **Bilateral Validity:** An argument of the form $\Gamma \vdash \Delta$ is *bilaterally valid*, relative to a set of admissible valuations V, $\Gamma \vDash_{B_V} \Delta$, just in case there is no $v \in V$ such that all of the stances in Γ are correct and all of the stances in Δ are incorrect.

Thus, for instance, $+\langle p\rangle, -\langle p\rangle \nvDash_{B_{\text{LP}}} +\langle q\rangle$ since there's a valuation admitted by LP in which both affirming p is correct and denying p is correct, but affirming q is incorrect, namely, one in which p is both true and false and q is just false. Likewise, $\nvDash_{B_{\text{K3}}} +p, -p$ since there's a valuation admitted by K3 in which both affirming p is incorrect and denying p is incorrect, namely, one in which p is neither true nor false. Neither $+p, -p \vDash_{B_{\text{FDE}}} +q$ nor $\vDash_{B_{\text{FDE}}} +p, -p$, since FDE admits both such valuations, whereas both $+p, -p \vDash_{B_{\text{CL}}} +q$ and $\vDash_{B_{\text{CL}}} +p, -p$, since CL admits valuations of neither sort.

This is a single intuitive notion of validity that applies to all logics in the FDE family. Indeed, it is simply a generalization of what Rumfitt [37, 224-225] [38, 808] calls "Smiley consequence." Now, I have defined this notion of validity, in the first instance, for multiple conclusion arguments. As far as a notion of validity for multiple conclusion arguments go, it seems to me to be as natural of a definition as one could have. In the special case of bilateral validity pertaining to single conclusion arguments, that $\Gamma \vdash A$ is valid means that, if all of the stances in Γ are correct, than A is also correct. Thus, given that the ND family is sound and complete with respect to single conclusion bilateral validity, we maintain a natural interpretation of the ND family as providing rules of *inference*, where, if A can be proven from Γ, that means that if you are correct in taking all of the stances in Γ, then you ought to take stance A.[18] Or, put with a slightly different modal flavor, $\Gamma \vdash A$ can be understood as saying that taking all of the stances in Γ *commits* one to taking the stance A.

As I show in the appendix, the bilateral proofs systems for the FDE family presented above are all sound and complete with respect to bilateral validity for each logic. Thus, we can define each logic, Bilateral CL, Bilateral LP, Bilateral K3, and Bilateral FDE in terms of their respective consequence relations that are realized both semantically, in the bilateral validities, and deductively, in the provable sequents.

6 Reasoning with the Liar

We are now in a position to return to formalize the informal reasoning about the liar presented at the beginning of this paper. Let us supplement our bilateral sequent systems with the following rules for the truth operator Tr such that affirming $Tr\ulcorner\varphi\urcorner$ is correct just in case affirming φ is correct and denying $Tr\ulcorner\varphi\urcorner$ is correct just in case denying φ is correct:[19]

[18] It's worth noting that single conclusion bilateral sequent calculi for these logics are also possible, but I will not pursue the development of them here.

[19] See Incurvati and J. Schlöder [20, 198-199] for bilateral natural deduction truth rules of the same sort. Just for simplicity, to keep things at the sentential level, I treat $Tr\ulcorner\cdot\urcorner$ as a unary sentential connective here rather than as 1-place predicate, as Ripley [35] [36] does, but everything said here goes for that alternative treatment as well.

$$\frac{\Gamma, +\langle\varphi\rangle \vdash \Delta}{\Gamma + \langle Tr\ulcorner\varphi\urcorner\rangle \vdash \Delta}\ +_{Tr_L} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle, \Delta}{\Gamma \vdash +\langle Tr\ulcorner\varphi\urcorner\rangle, \Delta}\ +_{Tr_R}$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash \Delta}{\Gamma - \langle Tr\ulcorner\varphi\urcorner\rangle \vdash \Delta}\ -_{Tr_L} \qquad \frac{\Gamma \vdash -\langle\varphi\rangle, \Delta}{\Gamma \vdash -\langle Tr\ulcorner\varphi\urcorner\rangle, \Delta}\ -_{Tr_R}$$

Let us further supplement our systems with sequent rules for affirming and denying the sentence λ, which says of itself that it's not true:

$$\frac{\Gamma, +\langle\neg Tr\ulcorner\lambda\urcorner\rangle \vdash \Delta}{\Gamma, +\langle\lambda\rangle \vdash \Delta}\ +_{\lambda_L} \qquad \frac{\Gamma \vdash +\langle\neg Tr\ulcorner\lambda\urcorner\rangle, \Delta}{\Gamma \vdash +\langle\lambda\rangle, \Delta}\ +_{\lambda_R}$$

$$\frac{\Gamma, -\langle\neg Tr\ulcorner\lambda\urcorner\rangle \vdash \Delta}{\Gamma, -\langle\lambda\rangle \vdash \Delta}\ -_{\lambda_L} \qquad \frac{\Gamma \vdash -\langle\neg Tr\ulcorner\lambda\urcorner\rangle, \Delta}{\Gamma \vdash -\langle\lambda\rangle, \Delta}\ -_{\lambda_R}$$

With these rules, we can reason as follows:

$$\begin{array}{c} \dfrac{\dfrac{\dfrac{+\langle\lambda\rangle \vdash +\langle\lambda\rangle}{+\langle\lambda\rangle \vdash +\langle Tr\ulcorner\lambda\rangle}\ +_{Tr_R}}{+\langle\lambda\rangle \vdash -\langle\neg Tr\ulcorner\lambda\urcorner\rangle}\ -_{\neg_R}}{\underline{+\langle\lambda\rangle \vdash -\langle\lambda\rangle}}\ -_{\lambda_R} \\ \vdots \end{array} \qquad \begin{array}{c} \dfrac{\dfrac{\dfrac{-\langle\lambda\rangle \vdash -\langle\lambda\rangle}{-\lambda \vdash -\langle Tr\ulcorner\lambda\rangle\urcorner}\ -_{Tr_R}}{-\langle\lambda\rangle \vdash +\langle\neg Tr\ulcorner\lambda\urcorner\rangle}\ +_{\neg_R}}{\underline{-\langle\lambda\rangle \vdash +\langle\lambda\rangle}}\ +_{\lambda_R} \\ \vdots \end{array}$$

Given that all the conclusions here are single, we can read $A \vdash B$ as saying that taking stance A commits one to taking stance B. The proof on the left reads as follows. Affirming the liar commits one to affirming the liar. So, affirming the liar commits one to affirming that the liar is true. Accordingly, affirming the liar commits one to denying that the liar is not true. But "that the liar is not true" is just what the liar says, and so affirming the liar commits one to denying the liar. The proof on the right reads as follows. Denying the liar commits one to denying the liar. So, denying the liar commits one to denying that the liar is true. Accordingly, denying the liar commits one to affirming that the liar is not true. But "that the liar is not true" is just what the liar says, and so denying the liar commits one to affirming the liar. All of this reasoning seems impeccable. Moreover, it seems like the sequents we end up with here express exactly what we want to say about the liar: the liar is a sentence such that affirming it commits one to denying it and denying it commits one to affirming it. There may be things one wants to say beyond that, but that much, it seems, is undeniable.[20]

The logical steps involved in the above reasoning—the instances of Reflexivity and the uses of the negation rules—are valid in all four bilateral systems in the FDE family. For the non-logical steps to be valid, we suppose that the semantics of our truth-predicate Tr is such that $v(Tr\ulcorner\varphi\urcorner)$ always equals $v(\varphi)$, and we suppose that $v(\lambda)$ always equals $v(\neg Tr\ulcorner\lambda\urcorner)$. Given this, it is easy to see that $v(\lambda)$ must be either $\varnothing$ or $\{1, 0\}$ in all FDE valuations. If $1 \in v(\lambda)$, then $1 \in v(Tr\ulcorner\lambda\urcorner)$, and so $0 \in v(\neg Tr\ulcorner\lambda\urcorner)$, and so $0 \in v(\lambda)$. Likewise if $0 \in v(\lambda)$, then $0 \in v(Tr\ulcorner\lambda\urcorner)$, and so $1 \in v(\neg Tr\ulcorner\lambda\urcorner)$, and so $1 \in v(\lambda)$. So, $v(\lambda)$ must be either $\varnothing$ or $\{1, 0\}$ in all FDE valuations. This means that it must be $\varnothing$ in all K3 valuations, it must be $\{1, 0\}$ in all LP valuations, and there is no value it can possibly take in any CL valuation.

Now, given that more inferences can be made in B_{K3} and B_{LP} than can be made in B_{FDE} in virtue of the presence of coordination principles, we can consider what further reasoning with the liar we can do in these two logics. In particular let us consider again the principles of Out and In:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, A^*}\ \text{Out} \qquad\qquad \frac{\Gamma \vdash A, \Delta}{\Gamma, A^* \vdash \Delta}\ \text{In}$$

Recall that Out is valid in B_{LP} but invalid in B_{K3} whereas In is valid in B_{K3} but invalid in B_{LP}. So, B_{K3} lets us reason from the conclusion of the above two proofs as follows:[21]

$$\dfrac{\dfrac{+\langle\lambda\rangle \vdash -\langle\lambda\rangle}{+\langle\lambda\rangle, +\langle\lambda\rangle \vdash}\ \text{In}}{+\langle\lambda\rangle \vdash}\ \text{Cont.} \qquad\qquad \dfrac{\dfrac{-\langle\lambda\rangle \vdash +\langle\lambda\rangle}{-\langle\lambda\rangle, -\langle\lambda\rangle \vdash}\ \text{In}}{-\langle\lambda\rangle \vdash}\ \text{Cont.}$$

We read these above two proofs as follows. Affirming the liar commits one to denying the liar. Accordingly, it's incoherent to affirming the liar. Likewise, denying the liar commits one to affirming the liar. Accordingly, it's incoherent

[20] It's important to emphasize that the consequence relation at play here is one of *commitment* preservation, rather than *entitlement* preservation. See Incurvati and J. Schlöder [21] on this point, who argue that the truth-rules here preserve commitment but not evidence.

[21] I show Contraction for clarity, but, technically, it is built into out treatment of what flanks the two sides of the turnstile as sets (rather than multi-sets or sequences).

to deny the liar. Thus, it's incoherent to affirm the liar, and it's incoherent to deny the liar. B_{LP} on the other hand, lets us reason as follows:

$$\dfrac{\dfrac{+\langle\lambda\rangle \vdash -\langle\lambda\rangle}{\vdash -\langle\lambda\rangle, -\langle\lambda\rangle}\text{ Out}}{\vdash -\langle\lambda\rangle}\text{ Cont.} \qquad \dfrac{\dfrac{-\langle\lambda\rangle \vdash +\langle\lambda\rangle}{\vdash +\langle\lambda\rangle, +\langle\lambda\rangle}\text{ Out}}{\vdash +\langle\lambda\rangle}\text{ Cont.}$$

According to B_{LP}, affirming the liar commits one to denying the liar, and denying the liar commits one to affirming the liar, and so one is committed both to denying the liar and to affirming the liar. So, whereas, in the case of B_{K3}, both of the two opposite stances towards the liar are incorrect, in the case of B_{LP}, both stances are correct.

Consider now (the multiple conclusion generalizations of) metainferential Explosion and Excluded Middle:

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash A^*, \Delta}{\Gamma \vdash \Delta}\text{ Explo.} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma, A^* \vdash \Delta}{\Gamma \vdash \Delta}\text{ Ex. Mid}$$

Once again, B_{K3} rejects Explosion, and good thing that it does, since, if Explosion were valid in B_{K3}, it could be applied at the end of the above proof to yield the empty sequent (and so every sequent, via Weakening). Likewise, B_{LP} rejects Excluded Middle, and, once again, good thing that it does, since Excluded Middle could be applied at the end of the above proof to yield the empty sequent (and so every sequent via Weakening).

These two non-classical systems make formally precise the two responses to the liar articulated in Section 2. A proponent of B_{K3} maintains the liar is such that it is never correct to either affirm it or deny it, whereas a proponent of B_{LP} maintains that the liar is such that it is always correct to both affirm it and deny it. Moreover, a proponent of B_{FDE} may acknowledge both of these possibilities but wish to stay neutral as to which is correct.[22]

7 On a Notable Fragment of B_{K3}

In order to appreciate the consequences of this account for contemporary debates surrounding the liar paradox, we first need to state some relations between

[22] See Beall [5] for some arguments for the weaker FDE position in a unilateral context, which are applicable here.

bilateral validity and unilateral validity. In general, unilateral validity, as it pertains to logics in the FDE family, says that $\Phi \vDash_V \Psi$ just in case there's no valuation in v such that all of the elements of Φ are true and all of the elements of Ψ are untrue. That is:

> **Unilateral Validity:** $\Phi \vDash_V \Psi$ just in case there is no $v \in V$ such $1 \in v(\varphi)$ for all $\varphi \in \Phi$ and $1 \notin v(\psi)$ for all $\psi \in \Psi$.

Where Φ is a set of sentences, I'll use the notation $+\langle\Phi\rangle$ as shorthand for $+\langle\varphi_1\rangle, +\langle\varphi_2\rangle \ldots + \langle\varphi_n\rangle$ for all $\varphi \in \Phi$. We can now state the following result:

Proposition 5: $+\langle\Phi\rangle \vDash_{B_V} +\langle\Psi\rangle$ just in case $\Phi \vDash_V \Psi$.

Proof: Immediate from the definitions. □

So, the positive fragments of the bilateral consequence relations for LP, K3, and FDE are each identical to the unilateral consequence relations for these respective logics. However, in the bilateral consequence relations of these logics, as we've defined them, there are sequents that aren't to be found in the unilateral consequence relations. For instance, not only does the K3 consequence relation contain $+\langle p \vee q\rangle, +\langle\neg p\rangle \vDash +\langle q\rangle$ and $+\langle p \vee q\rangle, +\langle\neg p\rangle, +\langle\neg q\rangle \vDash$, but it also contains $+\langle p \vee q\rangle, -\langle p\rangle \vDash +\langle q\rangle$ and $+\langle p \vee q\rangle, -\langle p\rangle, -\langle q\rangle \vDash$. It is perhaps not surprising that not only does B_{K3} contain the unilateral K3 consequence relation as its solely positive fragment, but it also contains the unilateral CL consequence relation as its solely left-sided fragment. That is:

Proposition 6: $+\langle\Phi\rangle, -\langle\Psi\rangle \vDash_{B_{\mathrm{K3}}}$ just in case $\Phi \vDash_{\mathrm{CL}} \Psi$.

Proof: For the left to right direction, suppose $+\langle\Phi\rangle, -\langle\Psi\rangle \vDash_{B_{\mathrm{K3}}}$ but $\Phi \nvDash_{\mathrm{CL}} \Psi$. Then is no $v \in \mathrm{K3}$ such that $1 \in v(\varphi)$ for all $\varphi \in \Phi$ and $0 \in v(\psi)$ for all $\psi \in \Psi$ but there is some $v \in \mathrm{CL}$ such that $1 \in v(\varphi)$ for all $\varphi \in \Phi$ and $1 \notin v(\psi)$ for all $\psi \in \Psi$. This CL valuation will be such that $v(\varphi) = \{1\}$ for all $\varphi \in \Phi$ and $v(\psi) = \{0\}$ for all $\psi \in \Psi$. But that's a K3 valuation in which the above condition holds. Contradiction. So, if $+\langle\Phi\rangle, -\langle\Psi\rangle \vDash_{B_{\mathrm{K3}}}$, then $\Phi \vDash_{\mathrm{CL}} \Psi$. For the right to left direction, suppose $\Phi \vDash_{\mathrm{CL}} \Psi$ but $+\langle\Phi\rangle, -\langle\Psi\rangle \nvDash_{B_{\mathrm{K3}}}$. Then there is no $v \in \mathrm{CL}$ such that $1 \in v(\varphi)$ for all $\varphi \in \Phi$ and

$1 \notin v(\psi)$ for all $\psi \in \Psi$ but there is some $v \in$ K3 such that $1 \in v(\varphi)$ for all $\varphi \in \Phi$ and $0 \in v(\psi)$ for all $\psi \in \Psi$. This K3 valuation will be such that that $v(\varphi) = \{1\}$ for all $\varphi \in \Phi$ and $v(\psi) = \{0\}$ for all $\psi \in \Psi$. But that's a CL valuation in which the above condition holds. Contradiction. So, if $\Phi \vDash_{\mathrm{CL}} \Psi$, then $+\langle\Phi\rangle, -\langle\Psi\rangle \vDash_{B_{\mathrm{K3}}}$. □

Dually, we have the following fact about the relationship between bilateral LP and unilateral CL:

Proposition 7: $\vDash_{B_{\mathrm{LP}}} -\langle\Phi\rangle, +\langle\Psi\rangle$ just in case $\Phi \vDash_{\mathrm{CL}} \Psi$.

Proof: Proceeds exactly analogously to the proof above. □

Though CL shows up in both $\mathrm{B_{K3}}$ and $\mathrm{B_{LP}}$ (as does everything, given their duality), there are philosophical reasons to focus our discussion on $\mathrm{B_{K3}}$'s embedding of unilateral CL. Given that $\mathrm{BS_{K3}}$ is sound and complete with respect to $\mathrm{B_{K3}}$ validity, Cut and Weakening are eliminable, and the formulas never travel across the turnstile, the following solely left-sided fragment of $\mathrm{BS_{K3}}$ (taking just the rules for conjunction, disjunction, and the conditional) is a sound and complete sequent system for unilateral CL:

$$\frac{}{\Gamma, A, A^* \vdash}\ \text{Inc.}$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash}{\Gamma, +\langle\neg\varphi\rangle \vdash}\ +_{\neg} \qquad\qquad \frac{\Gamma, +\langle\varphi\rangle \vdash}{\Gamma, -\langle\neg\varphi\rangle \vdash}\ -_{\neg}$$

$$\frac{\Gamma, +\langle\varphi\rangle, +\langle\psi\rangle \vdash}{\Gamma, +\langle\varphi \wedge \psi\rangle \vdash}\ +_{\wedge} \qquad\qquad \frac{\Gamma, -\langle\varphi\rangle \vdash \quad \Gamma, -\langle\psi\rangle \vdash}{\Gamma, -\langle\varphi \wedge \psi\rangle \vdash}\ -_{\wedge}$$

$$\frac{\Gamma, +\langle\varphi\rangle \vdash \quad \Gamma, +\langle\psi\rangle \vdash}{\Gamma, +\langle\varphi \vee \psi\rangle \vdash}\ +_{\vee} \qquad\qquad \frac{\Gamma, -\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma, -\langle\varphi \vee \psi\rangle \vdash}\ -_{\vee}$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash \quad \Gamma, +\langle\psi\rangle \vdash}{\Gamma, +\langle\varphi \to \psi\rangle \vdash}\ +_{\to} \qquad\qquad \frac{\Gamma, +\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma, -\langle\varphi \to \psi\rangle \vdash}\ -_{\to}$$

On the semantics proposed, each of these sequents of the form $\Gamma \vdash$ will be valid just in case there is no valuation such that all of the stances in Γ are correct. Thus, we might interpret any sequent of this form as saying that the position consisting in all of the affirmations and denials in Γ is, as Restall [34] puts it, "incoherent" or, as Ripley [35] puts it, "out of bounds." Thus, for instance, the solely left-sided instance of the axiom schema of Explosion, which I've labeled "Incoherence," says that, relative any set of stances Γ, affirming and denying some sentence φ is always out of bounds. Likewise, the positive negation rule says that if, relative to some set of stances Γ, denying φ is out of bounds then, relative to Γ, affirming $\neg\varphi$ is out of bounds.

The interpretation I've just given of this solely left-sided fragment of $\mathrm{BS}_{\mathrm{K3}}$ is just the bilateral interpretation of the (unilateral) multiple conclusion classical sequent calculus proposed by Restall and developed by Ripley. Indeed, this fragment of $\mathrm{BS}_{\mathrm{K3}}$ *just is* Ketonen's [23] multiple conclusion classical sequent calculus. Using X and Z (capital χ and ζ) for sets of sentences, Ketonen's sequent calculus, K, is the following:

K:

$$\overline{X, \varphi \vdash \varphi, Z}$$

$$\frac{X \vdash \varphi, Z}{X, \neg\varphi \vdash Z}\ \mathrm{L}_{\neg} \qquad \frac{X, \varphi \vdash Z}{X \vdash \neg\varphi, Z}\ \mathrm{R}_{\neg}$$

$$\frac{X, \varphi, \psi \vdash Z}{X, \varphi \wedge \psi \vdash Z}\ \mathrm{L}_{\wedge} \qquad \frac{X \vdash \varphi, Z \quad X \vdash \psi, Z}{X \vdash \varphi \wedge \psi, Z}\ \mathrm{R}_{\wedge}$$

$$\frac{X, \varphi \vdash Z \quad X, \psi \vdash Z}{X, \varphi \vee \psi \vdash Z}\ \mathrm{L}_{\vee} \qquad \frac{X \vdash \varphi, \psi, Z}{X \vdash \varphi \vee \psi, Z}\ \mathrm{R}_{\vee}$$

$$\frac{X \vdash \varphi, Z \quad X, \psi \vdash Z}{X, \varphi \rightarrow \psi \vdash Z}\ \mathrm{L}_{\rightarrow} \qquad \frac{X, \varphi \vdash \psi, Z}{X \vdash \varphi \rightarrow \psi, Z}\ \mathrm{R}_{\rightarrow}$$

It is straightforward to provide a one-to-one translation to show that this multiple conclusion unilateral sequent calculus is simply a notational variants of the soley left-sided bilateral sequent calculus shown above it. To translate a unilateral multiple conclusion sequent of the form $\Phi \vdash \Psi$ to a bilateral sequent of the form $\Gamma \vdash$ let $\Gamma = \{+\langle\varphi\rangle \mid \varphi \in \Phi\} \cup \{-\langle\psi\rangle \mid \psi \in \Psi\}$. Conversely, to translate a $\mathrm{BK}_{\mathrm{ls}}$ sequent of the form $\Gamma \vdash$ to a K sequent of the form $\Phi \vdash \Psi$ let $\Phi = \{\varphi \mid +\langle\varphi\rangle \in \Gamma\}$ and $\Psi = \{\psi \mid -\langle\psi\rangle \in \Gamma\}$. The point of note here is not just that $\mathrm{BS}_{\mathrm{K3}}$ contains K as its solely left-sided fragment. The point is that $\mathrm{BS}_{\mathrm{K3}}$ contains K in such a way that makes the bilateral interpretation of K, proposed by Restall and Ripley *explicit in the notation itself*. For Restall and Ripley, the role of the turnstile is simply to partition the set of assertions from the set of denials. In a bilateral system, the positive and negative signing of formulas does this job, and it does it much more perspicuously.

At this point, it's worth pointing out that, given that every $\mathrm{B_{K3}}$ consequence is a $\mathrm{B_{CL}}$ consequence, not only is unilateral CL contained in the solely left-sided fragment of $\mathrm{B_{K3}}$, but, of course, it's also contained in the solely left-sided fragment of $\mathrm{B_{CL}}$. So, having made Restall and Ripley's bilateral interpretation of unilateral consequence explicit in our bilateral notation, we can ask: which bilateral consequence relations are Restall and Ripley implicitly working with? The answer, I take it, is that Restall is implicitly working with Bilateral CL whereas Ripley is implicitly working with Bilateral K3. Let me explain.

Restall and Ripley both adopt a bilateralist reading of the unilateral multiple conclusion sequent calculus, taking it that $X \vdash Z$ can be understood as expressing that the position consisting in affirming everything in X and denying everything in Z is incoherent or out of bounds. The core difference between Restall and Ripley in their acceptance or rejection of certain structural principles. Most fundamentally they differ over the status of the following principle:

$$\frac{X, \varphi \vdash Y \quad X \vdash \varphi, Y}{X \vdash Y}\ \text{Unilateral Cut}$$

Now, Cut is generally thought of as a kind of transitivity principle, and, when $\vdash$ is understood as expressing consequence, properly so-called (as it is in the bilateral sequent calculi put forward here), it *is* a transitivity principle. However *unilateral* Cut, as explicated in *bilateralist* terms by Restall and Ripley, is really not a transitivity principle at all. Rather, it is, as Restall [34] puts it, a principle of *Extensibility*. It says, relative any position $+\langle X\rangle, -\langle Y\rangle$ consisting in affirming everything in X and denying everything in X, if affirming φ is out of bounds and denying φ is out of bounds, then $+\langle X\rangle, -\langle Y\rangle$ must itself be out of bounds. In other words, for every position Γ and every sentence φ, if Γ is coherent, then Γ must be coherently extensible to either $+\langle\varphi\rangle$ or $-\langle\varphi\rangle$. Translated into our much more perspicuous bilateral notation, this principle is the following:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash \quad \Gamma, -\langle\varphi\rangle \vdash}{\Gamma \vdash}\ \text{Extensibility}$$

And this just is a particular instance of (metainferential) Bilateral Excluded Middle: the case in which Δ is null. Since this is valid in $\mathrm{B_{CL}}$ and invalid in $\mathrm{B_{K3}}$, and Restall accepts it but Ripley rejects it, we might think of Restall as implicitly endorsing the solely left-sided version of $\mathrm{B_{CL}}$ and Ripley as implicitly endorsing the solely left-sided version of $\mathrm{B_{K3}}$. It's this latter characterization on which I want to focus here.

The core move of Ripley's solution to the liar paradox is to reject Extensibility. According to Ripley, relative to any position Γ, affirming the liar is out of bounds, and denying the liar is out of bounds. So, we'll always have $\Gamma, +\langle\lambda\rangle \vdash$ and $\Gamma, -\langle\lambda\rangle \vdash$. But we shouldn't be able to conclude from this that any position Γ is out of bounds. Concretely, rejecting Extensibility, one blocks the following proof at the final step:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, +\langle\lambda\rangle, -\langle\lambda\rangle \vdash}\,\text{Inc.}}{\Gamma, +\langle\lambda\rangle, -\langle Tr\ulcorner\lambda\rangle \vdash}\,-_{Tr_L}}{\Gamma, +\langle\lambda\rangle, +\langle\neg Tr\ulcorner\lambda\urcorner\rangle \vdash}\,+_{\neg_L}}{\Gamma, +\langle\lambda\rangle, +\langle\lambda\rangle \vdash}\,+_{\lambda_L}}{\Gamma, +\langle\lambda\rangle \vdash}\,\text{Contr.} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, -\langle\lambda\rangle, +\langle\lambda\rangle \vdash}\,\text{Inc.}}{\Gamma, -\lambda, +\langle Tr\ulcorner\lambda\rangle\urcorner \vdash}\,+_{Tr_L}}{\Gamma, -\langle\lambda\rangle, -\langle\neg Tr\ulcorner\lambda\urcorner\rangle \vdash}\,-_{\neg_L}}{\Gamma, -\langle\lambda\rangle, -\langle\lambda\rangle \vdash}\,-_{\lambda_L}}{\Gamma, -\langle\lambda\rangle \vdash}\,\text{Contr.}}{\Gamma \vdash}\,\text{Extens.}$$

Translating this into multiple conclusion unilateral notation it is easy to see that this is indeed Ripley's solution to the liar (cf. [35, 145]), but, in this bilateral notation, it's clear that this just is the solution of $\mathrm{B_{K3}}$ outlined above, but restricted to the left side of the turnstile. Extending Ripley's solution beyond the left side of the turnstile, we have formal principles not only codifying which sets of stances it's *incoherent* to adopt, but which sets of stances one is *committed* to adopting, given one's adoption of various other stances. For instance, a proponent of Ripley's solution can accept all of the FDE reasoning shown in the previous section that formally codifies how affirming the liar commits one to denying the liar and denying the liar commits one to affirming the liar. Indeed, one can maintain that this is precisely *why* one shouldn't take a stance, positive or negative, on the liar. If you take one, you've got to take the other, and then you've taken opposite stances towards a single sentence, and so your position is incoherent. Our explicitly bilateral system thus formally captures aspects of the informal reasoning—left out by Ripley's system—that makes Ripley's solution to the liar so intuitively plausible.

It is easy to see, given the definition of B_{K3} validity, that the solely left-sided fragment of B_{K3} just is the "non-transitive" logic ST. This recasting of ST in a bilateral context, however, has important implications for understanding Ripley's ST-based solution to the liar in the context of the debate over "non-classical" versus "substructural" approaches to paradox.[23] Ripley [35] claims that "the core of the advantages of the ST approach over K3T- and LPT-based approaches" is that it enables one to have a transparent truth-predicate while retaining all of classical logic. In a sense, this is true. All of *unilateral* classical logic is retained in this solution. In a deeper sense, however, the solution is not classical, since all of *bilateral* classical logic is not retained. For instance, even though BS_{K3} proves $-\langle\varphi \vee \neg\varphi\rangle \vdash$ (which, in Ripley's imperspicuous unilateral notation, would be written as $\vdash \varphi\vee\neg\varphi$), BS_{K3} does not prove $\vdash +\langle\varphi\vee\neg\varphi\rangle$. It makes sense, on this interpretation, why it should not: one should not affirm $\lambda \vee \neg\lambda$ if one wants to avoid affirming a contradiction, since, given disjunction elimination and the truth and liar rules, affirming $\lambda\vee\neg\lambda$ enables one to conclude $+\langle\lambda\wedge\neg\lambda\rangle$.[24] The only way to extend Ripley's solution to the liar to the right side of the turnstile is by rejecting bilateral classicality. So, while Ripley says "there is no need, from an ST-based perspective, ever to criticize (on logical grounds) any classically-valid inference," this is false insofar as we're considering inferences that are *bilaterally* classically valid. The same remarks apply for Ripley's advertisement of his approach as enabling one to have a workable material conditional. While it's true that BS_{K3} proves $-\langle\varphi \to \varphi\rangle \vdash$, it's not the case that $\vdash +\langle\varphi \to \varphi\rangle$.

On the flip side of things, Ripley takes his approach to be *substructural*. However, the consequence relation of Bilateral K3, along with all of the other logics presented here, is *fully structural*. Most importantly, the consequence

[23]For discussion of this debate, see Ripley [35], [36], Shaprio [39]

[24]The proof in the ND system goes as follows:

$$\dfrac{\dfrac{+\langle\lambda\vee\neg\lambda\rangle \quad \dfrac{}{+\langle\lambda\rangle}{}^{1} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{+\langle\neg\lambda\rangle}{}^{2}}{-\langle\lambda\rangle}}{-\langle Tr\ulcorner\lambda\urcorner\rangle}}{+\langle\neg Tr\ulcorner\lambda\urcorner\rangle}}{+\langle\lambda\rangle}}{+\langle\lambda\rangle} \qquad \dfrac{+\langle\lambda\vee\neg\lambda\rangle \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{+\langle\lambda\rangle}{}^{3}}{+\langle Tr\ulcorner\lambda\urcorner\rangle}}{-\langle\neg Tr\ulcorner\lambda\urcorner\rangle}}{-\langle\lambda\rangle}}{+\langle\neg\lambda\rangle} \quad \dfrac{}{+\langle\neg\lambda\rangle}{}^{4}}{+\langle\neg\lambda\rangle}}{+\langle\lambda\wedge\neg\lambda\rangle}$$

relation of Bialteral K3 is completely transitive. One cannot derive both $+\langle\varphi\rangle \vdash$ and $\vdash +\langle\varphi\rangle$, and so there is no reason to restrict Cut. Now, there might be other good reasons to reject transitivity.[25] However, from the perspective developed here, the liar paradox is not one of them.

8 Conclusion

I have put forward new bilateral systems for K3, LP, and FDE. Given that the rules of these proof systems are seperable, harmonious, and straightforwardly intuitively intelligible, they enable an inferentialist account of the meanings of the connectives that figure in these logics. One crucial upshot of this account is that the meanings of all of the connectives in all of these logics, understood in terms of the operational rules of governing their use, are the same. After all, all logics have the same operational rules—all that differs are the coordination principles. Another crucial upshot of this account is that it enables us to maintain what Incurvati and Schlöder [20], [21] call an "inferential deflationist" account of truth, according to which the meaning of truth is given entirely in the terms of rules governing assertions and denials stated above, while also acknowledging the possibility of truth-value gaps and gluts. Insofar as the meaning of truth is understood in this way, we can say that what it is for a sentence to be neither true nor false is for it to be such that we should neither assert it nor deny it. Likewise, what it is for a sentence to be both true and false is for it to be such that we should both assert and deny it. Having formally codified the rules governing assertions and denials to give sense to these dual notions in normative pragmatic terms, I take myself to have provided an inferential deflationist account of gaps and gluts. A "metaphysically lightweight" account of gluts in particular seems especially philosophically welcome. I want to conclude by considering and briefly responding to a potential "inferential expressivist" objection to this account.

According to inferential expressivism, recently developed by Incurvati and Schlöder [20], we can understand the meaning of expressions such as negation,

[25]See, for instance, Hlobil [17], Brandom [8], Simonelli [41], and Hlobil and Brandom [18], for reasons to reject transitivity in the context of modeling the sort of defeasible material inferences mentioned very briefly at the end of this paper.

conditionals, epistemic modals, and the truth-operator in terms of how their use inferentially relates to linguistic acts that play a fundamentally expressive role, such as assertion and denial. With Hlobil and Brandom [18], I take it that the expressive roles of assertion and denial are best understood in normative terms, and, in particular, as opposing "moves" in what Brandom [7] speaks of as "the game of giving and asking for reasons." Assertion is a basic move in the game, and denial is a basic counter-move, constituting a *challenge* to an assertion. In response to a challenge, one must *justify* one's assertion by citing *reasons* for it, reasons to which the challenger must respond with reasons of her own. Thus, successful acts of assertion are rationally underwritten by *implications*—reasons *for*—and successful acts of denial, which serve as challenges to assertions, are rationally underwritten by *incompatibilities*—reasons *against*. In this way denial can be understood as a "primitive operation that registers incompatibility," [20, 198], and, insofar as negation is directly inferentially linked to denial, it too can be understood as expressing incompatibility.

The thought that denial expresses incompatibility in this way might seem to be at odds with the thought that it is possible for there to be some sentences such that they ought to be both asserted and denied. After all, if denial expresses incompatibility, then isn't it the case that, for any sentence φ, $+\langle\varphi\rangle$ and $-\langle\varphi\rangle$ must be normatively incompatible in the sense that one act rationally rules out the other? Actually, no. I think not. I take it that it does not follow from the fact that denial expresses incompatibility that, for any sentence φ, $+\langle\varphi\rangle$ and $-\langle\varphi\rangle$ must be incompatible any more than it follows from the fact that birds fly that, for any bird x, x must fly.[26] My suggestion, then, is that we hear "denial expresses incompatibility," as it's used in the context of an inferential expressivist account of negation, like "birds fly": as a *generic* sentence, expressing a good but defeasible inference.[27] *In general*, denial expresses incompatibility, and the fact that it does is essential to our core concept of denial (just as the fact that that birds fly is essential to our core concept of a bird). However, this does not entail that *in every single case*, denial expresses incompatibility; there may be some odd cases where both asserting something and denying it is not rationally self-undermining but, rather, is just what one rationally ought to do. I do not

[26] Birds fly, but Mumble the penguin is a bird who doesn't fly.
[27] See Stovall [46] for an inferentialist account of generics along these lines.

myself know whether there really are any such odd cases, but, if there are, I take it that the case of the liar is one of them.

9 Appendix: Technical Results

9.1 Eliminibility of Non-Atomic Axioms, Cut, Weakening

Proposition 8: All axiom schemas of all BS systems can be restricted to atomics.

Proof: We consider Reflexivity first. The proof involves two inductions. For the first induction, we show that any sequent of the $\Gamma, A \vdash A, \Delta$, where A is atomic, is derivable from solely atomic axioms by induction on complexity of the most complex formulas in Γ, Δ. This is trivial, as we always retain A on both sides through any application of the connective rules. For the second induction, we show that any sequent of the form $\Gamma, A \vdash A, \Delta$ is derivable by induction on the complexity of A. The base case is established by the first induction. For the inductive step, we suppose that A is complexity $n + 1$ and show we can derive $\Gamma, A \vdash A$ from some number of sequents of the form $\Gamma', B \vdash B$ where B is complexity n. Where A is of the form $\boldsymbol{c}\langle\varphi \circ \psi\rangle$ the following derivation establishes this:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \boldsymbol{a}, \langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{a}\langle\varphi\Delta}\ \text{Reflex.} \qquad \dfrac{}{\Gamma, \boldsymbol{a}, \langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{b}\langle\psi\rangle, \Delta}\ \text{Reflex.}}{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle, \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{R}}}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle, \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{L}}}$$

The case where A is of the form $\boldsymbol{c}^*\langle\varphi \circ \psi\rangle$ is exactly dual.

Now, consider Explosion. The first induction is exactly the same. For the inductive step of the second induction, we show that we can derive $\Gamma, A, A^* \vdash \Delta$ from some number of sequents of the form $\Gamma', B, B^* \vdash \Delta$. There is just one case to consider:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \boldsymbol{a}, \langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle, \boldsymbol{a}^*\langle\varphi\rangle \vdash \Delta}\ \text{Explo.} \qquad \dfrac{}{\Gamma, \boldsymbol{a}, \langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle, \boldsymbol{b}^*\langle\psi\rangle \vdash \Delta}\ \text{Explo.}}{\Gamma, \boldsymbol{a}, \langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{L}}}}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{L}}}$$

Excluded middle is exactly dual. □

Proposition 9: Cut is eliminable in all BS systems.

Proof: The generalized Cut Elimination proofs proceeds similarly to the generalized proof given by Simonelli [42] in the context of classical logic. Accordingly, I will only sketch the basic proof strategy and show the crucial case. The proof is a double induction with a primary induction is on Cut formula weight and a secondary induction on Cut height. We show first by induction on Cut height that Cut on atomics is eliminable. To do this, we show, first, that Cut on axioms is elinimable, and, second, that, in all applications of Cut where Cut formula is not principal (where, if it's atomic, it never will be), the application of Cut be pushed up the proof tree. This induction then serves as our base case for the primary induction on formula weight. For the inductive step, we show that either Cut height can be reduced, or, in the crucial cases where the Cut formula is principal in both premises, the weight of the Cut formula can be reduced. For the $\boldsymbol{c}$ rules, we have the following reduction:

$$\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \Delta}\,\boldsymbol{c}_{\circ_L} \qquad \dfrac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle, \Delta \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle, \Delta}\,\boldsymbol{c}_{\circ_R}}{\Gamma \vdash \Delta}\,\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \Delta \quad \Gamma \vdash \boldsymbol{a}\langle\varphi\rangle, \Delta}{\Gamma, \boldsymbol{b}\langle\psi\rangle \vdash \Delta}\,\text{Cut} \qquad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle, \Delta}{\Gamma \vdash \Delta}\,\text{Cut}$$

The reduction for the $\boldsymbol{c}^*$ rules is exactly dual. □

Remark: Following the approach of Simonelli [42], based on that of Hacking [16] and Kremer [25], we can maintain that the proofs of propositions 8 and 9 establish the *harmony* between the positive and negative and left and right rules in the BS systems.

Proposition 10: Weakening is eliminable in all BS systems.

Proof: Weakening can be derived directly from Cut, given Reflexivity.[28]

[28] A direct proof of height-preserving Weakening elimination can be given (cf. Negri and von Plato [28]). I present this simple proof using Cut just for ease of proof.

Where Δ has n signed sentences, we derive Weakening on the left as follows:

$$\dfrac{\Gamma \vdash \Delta \quad \overline{\Gamma, \Gamma', \Delta \vdash \Delta}^{\text{ Reflex.}}}{\Gamma, \Gamma' \vdash \Delta}\ n \text{ applications of Cut}$$

Weakening on the right is derived similarly. Since Weakening is eliminable given Cut, and Cut is eliminable, Weakening is eliminable.

9.2 Generalized Soundness and Completeness

Proposition 11: *Soundness of BS_{FDE}:* If $\mathrm{BS}_{\mathrm{FDE}}$ proves $\Gamma \vdash \Delta$, then $\Gamma \vDash_{\mathrm{FDE}} \Delta$

Proof: Straightforward by induction. For the base case, any instance of the axiom schema $\Gamma, A \vdash A, \Delta$ is valid, since, for any valuation, if A is correct, then A is not incorrect. For the inductive step, we show that our rules preserve validity. The case of negation is obvious. For the binary connective schema, suppose we have $\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \Delta$ at height n and derive $\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \Delta$ at height $n+1$. By our inductive hypothesis, there's no valuation such that all the stances in Γ are correct, $[\boldsymbol{a}] \in v(\varphi)$, $[\boldsymbol{b}] \in v(\psi)$, and all of the stances in Δ are incorrect. Since $[\boldsymbol{c}] \in v(\varphi \circ \psi)$, just in case $[\boldsymbol{a}] \in v(\varphi)$, and $[\boldsymbol{b}] \in v(\psi)$ there's no valuation such that all the stances in Γ are correct, $[\boldsymbol{c}] \in v(\varphi \circ \psi)$, and all of the stances in Δ are incorrect. So, if $\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vDash \Delta$, then $\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vDash \Delta$. The other cases are similar. $\square$

Proposition 12: *Completeness of BS_{FDE}:* If $\Gamma \vDash_{\mathrm{B_{FDE}}} \Delta$, then $\mathrm{BS}_{\mathrm{FDE}}$ proves $\Gamma \vdash \Delta$.

Proof: We prove the contrapositive. Suppose $\Gamma \vdash \Delta$ is not provable. We will consider a *reduction tree* that extends $\Gamma \vdash \Delta$ in a number of steps using the following procedure:[29]

1. If $+\langle\neg\varphi\rangle \in \Gamma$ and $-\langle\varphi\rangle \notin \Gamma$, extend $\Gamma \vdash \Delta$ to $\Gamma, -\langle\varphi\rangle \vdash \Delta$.
2. If $-\langle\neg\varphi\rangle \in \Gamma$ and $+\langle\varphi\rangle \notin \Gamma$, extend $\Gamma \vdash \Delta$ to $\Gamma, +\langle\varphi\rangle \vdash \Delta$
3. If $+\langle\neg\varphi\rangle \in \Delta$ and $-\langle\varphi\rangle \notin \Delta$, extend $\Gamma \vdash \Delta$ to $\Gamma \vdash -\langle\varphi\rangle, \Delta$.

[29]This method of reduction trees is deployed by Ripley [35] for ST, based on Takeuti [47]. A very similar method involves the construction of saturated sets, spelled out in detail in [22, 32-35]. See [45] for a proof of this sort for signed tableuex using a similar schematization strategy to the one deployed here.

4. If $-\langle \neg\varphi \rangle \in \Delta$ and $+\langle \varphi \rangle \notin \Delta$, extend $\Gamma \vdash \Delta$ to $\Gamma \vdash +\langle \varphi \rangle, \Delta$

5. If $\boldsymbol{c}\langle \varphi \circ \psi \rangle \in \Gamma$, and either $\boldsymbol{a}\langle \varphi \rangle \notin \Gamma$ or $\boldsymbol{b}\langle \psi \rangle \notin \Gamma$, extend $\Gamma \vdash \Delta$ to $\Gamma, \boldsymbol{a}\langle \varphi \rangle, \boldsymbol{b}\langle \psi \rangle \vdash \Delta$

6. If $\boldsymbol{c}^*\langle \varphi \circ \psi \rangle \in \Gamma$ and neither $\boldsymbol{a}^*\langle \varphi \rangle \in \Gamma$ nor $\boldsymbol{b}^*\langle \psi \rangle \in \Gamma$, extend $\Gamma \vdash \Delta$ into two branches: $\Gamma, \boldsymbol{a}^*\langle \varphi \rangle \vdash \Delta$ and $\Gamma, \boldsymbol{b}^*\langle \psi \rangle \vdash \Delta$ and continue the procedure on each.

7. If $\boldsymbol{c}\langle \varphi \circ \psi \rangle \in \Delta$ and neither $\boldsymbol{a}\langle \varphi \rangle \in \Delta$ nor $\boldsymbol{b}\langle \psi \rangle \in \Delta$, extend $\Gamma \vdash \Delta$ into two branches: $\Gamma \vdash \boldsymbol{a}\langle \varphi \rangle, \Delta$ and $\Gamma \vdash \boldsymbol{b}\langle \psi \rangle, \Delta$ and continue the procedure on each.

8. If $\boldsymbol{c}^*\langle \varphi \circ \psi \rangle \in \Delta$, and either $\boldsymbol{a}^*\langle \varphi \rangle \notin \Delta$ or $\boldsymbol{b}^*\langle \psi \rangle \notin \Delta$, extend $\Gamma \vdash \Delta$ to $\Gamma \vdash \boldsymbol{a}^*\langle \varphi \rangle, \boldsymbol{b}^*\langle \psi \rangle, \Delta$

Since $\Gamma \vdash$ contains a finite number of formulas of finite length, there is a finite number of steps we must take until this procedure terminates.

Now, suppose we have constructed a reduction tree for $\Gamma \vdash \Delta$. The are two possibilities. The first possibility is that each of the final sequents in the reduction tree is of the form $\Gamma', A \vdash A, \Delta'$ for some atomic formula A. In this case, $\Gamma \vdash \Delta$ is provable. To see this, note that each of these final sequents is an instance of the axiom schema, and now see that we can now run the extension procedure we just did in reverse to arrive at a proof of $\Gamma \vdash \Delta$. Contradiction, so there must be some final sequent $\Gamma' \vdash \Delta'$ in reduction tree of $\Gamma \vdash \Delta$ that such that there is no atomic formula A such that $A \in \Gamma$ and $A \in \Delta$.

We now show by induction on formula complexity that there is an FDE valuation v such that

if $+\langle \varphi \rangle \in \Gamma'$, then $1 \in v(\varphi)$

if $-\langle \varphi \rangle \in \Gamma'$, then $0 \in v(\varphi)$

if $+\langle \varphi \rangle \in \Delta'$, then $1 \notin v(\varphi)$

if $-\langle \varphi \rangle \in \Delta'$, then $0 \notin v(\varphi)$

For the base case, we define an FDE valuation v such that, for all atomics p, $1 \in v(p)$ just in case $+\langle p \rangle \in \Gamma'$, $0 \in v(p)$ just in case $-\langle \varphi \rangle \in \Gamma'$, $+\langle p \rangle \in \Delta'$ just in case $1 \notin v(p)$, and $-\langle p \rangle \in \Delta'$ just in case $0 \notin v(p)$. Since there is no atomic signed formula A such that $A \in \Gamma$ and $A \in \Delta$, there exists such a valuation.

For the inductive step, we suppose the above condition holds for formulas of complexity n and show that it holds for formulas of complexity $n + 1$.

Consider first the case in which φ is of the form $\neg\psi$. If $+\langle\neg\psi\rangle \in \Gamma'$, then, by rule 1 of the reduction procedure, $-\langle\psi\rangle \in \Gamma'$. By the inductive hypothesis, $0 \in v(\psi)$, and so $1 \in v(\neg\psi)$. The same reasoning goes if $-\langle\neg\psi\rangle \in \Gamma'$. Suppose now that $+\langle\neg\psi\rangle \in \Delta'$. By Rule 3 of the reduction procedure $-\langle\psi\rangle \in \Delta'$. By the inductive hypothesis $0 \notin v(\psi)$, and so $1 \notin v(\neg\psi)$. The same reasoning goes if $-\langle\neg\psi\rangle \in \Delta'$.

Now suppose that φ is of the form $\psi \circ \chi$, for any binary connective $\circ$. Here, to establish the above condition for any connective, we will show, for any stance $\boldsymbol{c}$:

if $\boldsymbol{c}\langle\varphi\rangle \in \Gamma'$, then $[\boldsymbol{c}] \in v(\varphi)$

if $\boldsymbol{c}^*\langle\varphi\rangle \in \Gamma'$, then $[\boldsymbol{c}^*] \in v(\varphi)$

if $\boldsymbol{c}\langle\varphi\rangle \in \Delta'$, then $[\boldsymbol{c}] \notin v(\varphi)$

if $\boldsymbol{c}^*\langle\varphi\rangle \in \Delta'$, then $[\boldsymbol{c}^*] \notin v(\varphi)$

Given that $\boldsymbol{c}$ is either $+$ or $-$, and whichever one it is, $\boldsymbol{c}^*$ is the other, establishing this suffices to establish the above condition. So, there are four cases to consider. Suppose first $\boldsymbol{c}\langle\psi \circ \chi\rangle \in \Gamma'$. Then, by rule 5 of the reduction procedure, $\boldsymbol{a}\langle\psi\rangle \in \Gamma'$ and $\boldsymbol{b}\langle\chi\rangle \in \Gamma'$. By the inductive hypothesis $[\boldsymbol{a}] \in v(\psi)$ and $[\boldsymbol{b}] \in v(\chi)$, and so $[\boldsymbol{c}] \in v(\psi \circ \chi)$. Suppose now $\boldsymbol{c}\langle\psi \circ \chi\rangle \in \Delta'$. By rule 7 of the reduction procedure, either $\boldsymbol{a}\langle\psi\rangle \in \Delta'$ or $\boldsymbol{b}\langle\chi\rangle \in \Delta'$. By the inductive hypothesis, if $\boldsymbol{a}\langle\psi\rangle \in \Delta'$, then $[\boldsymbol{a}^*] \notin v(\psi)$ and, if $\boldsymbol{b}\langle\chi\rangle \in \Delta'$, then $[\boldsymbol{b}^*] \notin v(\chi)$, and so, by the valuation function, $[\boldsymbol{c}] \notin v(\psi \circ \chi)$. The cases where $\boldsymbol{c}^*\langle\psi \circ \chi\rangle \in \Gamma'$ or $\boldsymbol{c}^*\langle\psi \circ \chi\rangle \in \Delta'$ are similar.

So, there is a valuation v such that $1 \in v(\varphi)$ for all formulas of the form $+\langle\varphi\rangle \in \Gamma'$ and $0 \in v(\varphi)$ for all formulas of the form $-\langle\varphi\rangle \in \Gamma'$, $1 \notin v(\varphi)$ for all formulas of the form $+\langle\varphi\rangle \in \Delta'$ and $0 \notin v(\varphi)$ for all formulas of the form $-\langle\varphi\rangle \in \Delta'$. That is, there is a valuation v such that all of the stances in Γ' are correct and all of the stances in Δ' are incorrect. That is, $\Gamma' \nvDash \Delta'$. But $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$, so this is true of Γ and Δ as well. Thus, $\Gamma \nvDash \Delta$. □.

Proposition 13: *Soundness and Completeness for BS_{K3}*: $\Gamma \vDash_{\mathrm{B}_{\mathrm{K3}}} \Delta$ just in case $\mathrm{BS}_{\mathrm{K3}}$ proves $\Gamma \vdash \Delta$.

Soundness: Same as soundness for BS_{FDE} except we also note that any instance $\Gamma', A, A^* \vdash \Delta$ is valid, since there is can be no K3 valuation where A and A^* are correct, since K3 valuations cannot contain $\{1, 0\}$. □

Completeness: Same as completeness BS_{FDE} except that we also consider the case in which each of the final sequents in the reduction tree is of the form $\Gamma', A \vdash A, \Delta'$ or $\Gamma', A, A^* \vdash \Delta$ for some atomic formula A. Once again, if this is so, then $\Gamma \vdash \Delta$ would be provable, and so there must be some final sequent $\Gamma' \vdash \Delta'$ in reduction tree of $\Gamma \vdash \Delta$ that such that, for all atomic formulas A, neither $A \in \Gamma'$ and $A \in \Delta'$ nor $A, A^* \in \Gamma'$. Then, for the base case of the induction, we know that there is a K3 valuation v such that, for all atomics p, $1 \in v(p)$ just in case $+\langle p \rangle \in \Gamma'$ and $0 \in v(p)$ just in case $-\langle p \rangle \in \Gamma'$, since we know that it can't be the case that $+\langle p \rangle \in \Gamma'$ and $-\langle p \rangle \in \Gamma'$. Everything else proceeds the same. □

Proposition 14: *Soundness and Completeness for BS_{LP}*: $\Gamma \vDash_{B_{LP}} \Delta$ just in case BS_{LP} proves $\Gamma \vdash \Delta$.

Tweaks to the soundness and completeness proof for BS_{FDE} are exactly analogous to those made for BS_{K3} □

Proposition 15: All BN systems are sound and complete.

Soundness: Follows from the derivability of all of the rules in the BS systems.

Completeness: Given the mapping between these ND systems and Priest's, Priest's completeness proofs can be straightforwardly generalized in a similar fashion as above. □.

References

[1] A. R. Anderson and N. D. Belnap Jr. First degree entailments. *Math. Annalen*, 149:302–319, 1963.

[2] F. G. Asenjo. A calculus for antinomies. *Notre Dame Journal of Formal Logic*, 16(1):103–105, 1966.

[3] Sara Ayhan. Uniqueness of logical connectives in a bilateralist setting. In

Martin Blicha and Igor Sedlár, editors, *The Logica Yearbook 2020*, pages 1–16. College Publications, 2021.

[4] Jc Beall. Multiple-conclusion lp and default classicality. *Review of Symbolic Logic*, 4(2):326–336, 2011.

[5] Jc Beall. There is no logical negation: True, false, both, and neither. *Australasian Journal of Logic*, 14(1):1–29, 2017.

[6] Jc Beall, Michael Glanzberg, and David Ripley. *Formal Theories of Truth*. Oxford University Press, Oxford, 2018.

[7] Robert Brandom. *Making It Explicit: Reasoning, Representing, and Discursive Commitment*. Harvard University Press, Cambridge, Mass., 1994.

[8] Robert Brandom. From logical expressivism to expressivist logic: Sketch of a program and some implementations1. *Philosophical Issues*, 28(1):70–88, 2018.

[9] Pedro del Valle-Inclan. Harmony and normalisation in bilateral logic. *Bulletin of the Section of Logic*, 52(3):377–409, 2023.

[10] Pedro del Valle-Inclan and Julian J. Schlöder. Coordination and harmony in bilateral logic. *Mind*, 132(525):192–207, 2023.

[11] Michael Dummett. *The Logical Basis of Metaphysics*. Harvard University Press, Cambridge, Mass., 1991.

[12] J. Michael Dunn. Intuitive semantics for first-degree entailments and 'coupled trees'. *Philosophical Studies*, 29(3):149–168, 1976.

[13] Hartry H. Field. *Saving Truth From Paradox*. Oxford University Press, New York, 2008.

[14] Andreas Fjellstad. Non-classical elegance for sequent calculus enthusiasts. *Studia Logica*, 105(1):93–119, 2017.

[15] Gerhard Gentzen. Untersuchungen Über das logische schließen. i. *Mathematische Zeitschrift*, 35:176–210, 1935.

[16] Ian Hacking. What is logic? *Journal of Philosophy*, 76(6):285–319, 1979.

[17] Ulf Hlobil. A nonmonotonic sequent calculus for inferentialist expressivists. In Pavel Arazim and Michal Danč'ak, editors, *The Logica Yearbook 2015*, pages 87–105. College Publications, 2016.

[18] Ulf Hlobil and Robert Brandom. *Reasons for Logic, Logic for Reasons: Pragmatics, Semantics, and Conceptual Roles*. Routledge, New York, Forthcoming.

[19] Luca Incurvati and Julian Schlöder. Weak rejection. *Australasion Journal of Philosophy*, 95:741–760, 2017.

[20] Luca Incurvati and Julian J. Schlöder. *Reasoning with Attitude*. Oxford University Press USA, New York, 2023.

[21] Luca Incurvati and Julian J. Schlöder. Inferential deflationism. *The Philosophical Review*, forthcoming.

[22] Andrzej Indrzejczak. *Sequents and Trees: An Introduction to the Theory and Applications of Propositional Sequent Calculi*. 2021.

[23] Oiva Ketonen. Untersuchungen zum prädikatenkalkül. *Annales Academiae Scientiarum Fennicae Series A, I. Mathematica-physica*, 1944.

[24] Michael Kremer. Kripke and the logic of truth. *Journal of Philosophical Logic*, 17(3):225–278, 1988.

[25] Michael Kremer. Logic and meaning: The philosophical significance of the sequent calculus. *Mind*, 97(385):50–72, 1988.

[26] Saul Kripke. Outline of a theory of truth. *Journal of Philosophy*, 72(19):690–716, 1975.

[27] Nils Kürbis. Some comments on ian rumfitt's bilateralism. *Journal of Philosophical Logic*, 45(6):623–644, 2016.

[28] Sara Negri and Jan von Plato. *Structural Proof Theory*. Cambridge University Press, New York, 2001.

[29] C. S. Peirce. Oh the algebra of logic. *American Journal of Mathematics*, 3(1):15–57, 1880.

[30] Graham Priest. The logic of paradox. *Journal of Philosophical Logic*, 8(1):219–241, 1979.

[31] Graham Priest. *In Contradiction: A Study of the Transconsistent*. Oxford University Press, New York, 1987.

[32] Graham Priest. *Doubt Truth to Be a Liar*. Oxford University Press, New York, 2006.

[33] Graham Priest. Natural deduction systems for logics in the fde family. In *New Essays on Belnap-Dunn Logic*, pages 279–292. Springer, 2019.

[34] Greg Restall. Multiple conclusions. In Petr Hájek, Luis Valdés-Villanueva, and Dag Westerstahl, editors, *Logic, Methodology and Philosophy of Science*. College Publications, 2005.

[35] David Ripley. Paradoxes and failures of cut. *Australasian Journal of Philosophy*, 91(1):139–164, 2013.

[36] David Ripley. Revising up. *Philosophers' Imprint*, 13(5):1–13, 2013.

[37] Ian Rumfitt. The categoricity problem and truth-value gaps. *Analysis*, 57(4):223–236, 1997.

[38] Ian Rumfitt. "yes" and "no". *Mind*, 109:781–823, 2000.

[39] Lionel Shapiro. The very idea of a substructural approach to paradox. *Synthese*, 199(3):767–786, 2016.

[40] Lionel Shapiro. Lp, k3, and fde as substructural logics. In Pavel Arazim and Tomávs Lavička, editors, *The Logica Yearbook 2016*. College Publications, 2017.

[41] Ryan Simonelli. Considering the exceptions: On the failure of cumulative transitivity for indicative conditionals. *Synthese*, 200(5):1–21, 2022.

[42] Ryan Simonelli. A general schema for bilateral proof rules. *Journal of Philosophical Logic*, 2024.

[43] Timothy Smiley. Rejection. *Analysis*, 56:1–9, 1996.

[44] Timothy Smiley and Graham Priest. Can contradictions be true? *Aristotelian Society Supplementary Volume*, 67(1):17–54, 1993.

[45] Raymond M. Smullyan. *First-Order Logic*. Springer Verlag, New York, 1968.

[46] Preston Stovall. Characterizing generics are material inference tickets: A proof-theoretic analysis. *Inquiry*, (5):668–704, 2019.

[47] Gaisi Takeuti. *Proof Theory*. Elsevier Science Pub. Co., New York, 1975.

[48] Heinrich Wansing. Falsification, natural deduction and bi-intuitionistic logic. *Journal of Logic and Computation*, advance access:1–26, 2013.

[49] Henrich Wansing and Sara Ayhan. Logical multilateralism. *Journal of Philosophical Logic*, 2023.

[50] Stefan Wintein. On the strict-tolerant conception of truth. *Australasian Journal of Philosophy*, 92(1):1–20, 2014.

[51] Stefan Wintein. On all strong kleene generalizations of classical logic. *Studia Logica*, 104(3):503–545, 2016.

[52] Richard Zach. Cut elimination and normalization for generalized single and multi-conclusion sequent and natural deduction calculi. *Review of Symbolic Logic*, 14(3):645–686, 2021.