Bringing Bilateralisms Together:
A Unified Framework for Inferentialists

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0 Introduction

Inferentialism aims to account for the meanings of linguistic expressions in terms of the inferential rules governing their use. One of the main formal developments in the inferentialist program over the past few decades has been bilateralism in proof-theoretic semantics, according to which affirmation and denial are taken to be equally basic in providing an account of the meanings of expressions in terms of the rules governing their use in proof systems. Bilateralists are split, however, as to what this actually entails. Some bilateralists, following Greg Restall, use bilateralism to interpret existing proof systems, such as Gentzen's multiple conclusion classical sequent calculus. Other bilateralists, following Ian Rumfitt, develop distinctively bilateral proof systems in which formulas are positively or negatively signed. In this paper, I explore the respective virtues of these two styles of bilateralism in the context of the broader inferentialist program. I argue that, considered in this application, both forms of bilateralism have distinctive virtues. On the one hand, the multiple conclusion sequent systems readily made sense of by Restall's bilateralism are particularly well-suited to accommodating non-logical axioms encoding defeasible material inferential relations. On the other hand, the single conclusion bilateral sequents of the sort that figure in Rumfitt-style systems are particularly well-suited for formalizing a normative pragmatic theory of the sort developed by Robert Brandom. After laying out these respective virtues, I show how these forms of bilateralism can be brought together in a single bilateral system that has the virtues of both. While this formal bridge between bilateralisms is itself of significant interest to bilateralists of different stripes, the main upshot of this paper is a new bilateral system that is uniquely suited for inferentialist semantics.

1 Two Ways to Be a Bilateralist

A bilateral conception of a logic takes as basic two opposite ways of being related to the propositions expressed by the sentences with which the logic is concerned. On the standard way of thinking about these two ways, they are saying "Yes" to the question of whether the proposition is true or saying "No" to it, affirming the proposition or denying it, accepting it or rejecting it.[1] Little hangs on the exact vocabulary one prefers to use here. What is important is that there are these two opposite stances that one might take towards a proposition, a positive stance and a negative stance, and taking the negative stance towards some proposition is conceived as *distinct from* though *logically equivalent to* taking the positive stance towards its negation. With the position construed, prominent bilateralists include Greg Restall, David Ripley, Ian Rumfitt, Nassim Francez, and Luca Incurvati and Julian Schlöder. Dividing these bilateralists, however, are two very different ways to be a bilateralist, two distinct styles of bilateralism.

The first style of bilateralism, put foward by Restall (2005, 2009, 2013) and influentially developed by Ripley (2014, 2015, 2017), takes existing proof systems, such as Gentzen's sequent calculus for classical logic, and interprets them in a bilateralist fashion.[2] What is notable about Gentzen's classical se-

[1] One might also endorse a form of bilateralism, as Fine (2017) does, according to which these two opposite ways of being related to a proposition are not two ways that *we* might stand to it, but two opposite ways that *the world* might stand to it: verifying it or falsifying it. What is crucial about any sort of bilateral conception of logic is that two ways can be understood as opposites in the sense that one is a *positive* way of being related to a proposition and one is a *negative* way of being related to a proposition. Here, we will follow the main way form of bilateralism by thinking of these two ways as two ways *we* might be related to a proposition, but much if not all of what we say here will carry over to this alternate way of thinking about bilateralism, according to which the "two ways" are two ways that the world might stand with respect to proposition. For a precise account of an isomorphism between Fine's truth-maker bilateralism and Restall and Ripley's normative bilateralism, see Hlobil (2023).

[2] For other notable proponents and developers of this style of bilateralism, see Tanter (2021a, 2021b), Rosenblatt (2019), Hlobil (2019, 2023), Hlobil and Brandom (forthcoming), Francez

quent calculus is that the sequents one manipulates through the use of the calculus have *multiple conclusions*. The standard way of thinking about a multiple conclusion sequent of the form $\Gamma \vdash \Delta$, where Γ and Δ are both sets of sentences, is that the elements of Γ, collected conjunctively, *imply* the elements of Δ, collected disjunctively.[3] Critics of multiple conclusion sequent calculi have argued that "arguments" with multiple conclusions, understood in this way, are too far from our ordinary practices of reasoning to be appealed to in providing a proof-theoretic account of the meanings of the logical connectives.[4] In response to these concerns, Restall proposes a reading of multiple conclusion sequents according to which the turnstile plays the role not of separating *premises* from *conclusions* but of separating *affirmations* from *denials*. On this reading, a sequent of the form $\Gamma \vdash \Delta$ says that the position consisting in affirming everything in Γ and denying everything in Δ is incoherent or "out of bounds." This interepretation enables us to explicate and justify the standard structural and operational rules of classical logic in an intuitive way. For instance, we can understand the axiom schema of Identity, $\Gamma, A \vdash A, \Delta$, as telling us that it's always out of bounds to affirm and deny a single sentence A, no matter what else we affirm or deny. For operational rules, consider the negation rules of the classical sequent calculus:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \mathrm{L}_{\neg} \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \mathrm{R}_{\neg}$$

On the bilateralist interpretation, the left rule says that if the position consisting in affirming everything in Γ, denying everything in Δ, and denying A is out of bounds, then the position consisting in affirming everything in Γ, denying everything in Δ, and affirming $\neg A$ is out of bounds. The right rule says that if the position consisting in affirming everything in Γ, denying everything in Δ, and affirming A is out of bounds, then the position consisting in affirming everything in Γ, denying everything in Δ, and denying $\neg A$ is out of bounds. So, understanding the significance of speech acts in terms of their contribution

(2019), and Hjortland (2014).

[3]See Shoesmith and Smiley (1978) for the classical articulation of this standard interpretation.

[4]See, for instance, Rumfitt (2000, 2008) and Steinberger (2011) for criticisms of this sort.

to the (in)coherence of positions, this rule tells us that an affirmation of $\neg A$ has the same significance as a denial of A, and a denial of $\neg A$ has the same significance as an affirmation of A. In the same way, all of the other rules of a multiple conclusion sequent system like Gentzen's classical sequent calculus, LK, can be intuitively explained. In this way, Restall's bilateralism presents itself as a way of vindicating classical logic.

The second style of bilateralism, put foward Rumfitt (2000) and influentially developed by Francez (2014, 2015) and Incurvati and Schlöder (2017, 2019, 2023), involves constructing proof systems in a distinctively bilateralist manner, providing rules for manipulating positively or negatively signed formulas.[5] Dummett (1991) famously argues that, if we want to think of the meanings of the logical connectives in terms of the rules governing their use in proofs in natural deduction, we should be intuitionists rather than classicalists, since it is intuitionistic natural deduction rather than classical natural deduction that displays that proof-theoretic virtue of *harmony*, with the introduction and elimination rules fitting together as they ought, with each set of rules being neither too strong nor too weak relative to the other.[6] In response to Dummett, Rumfit (2000), drawing on prior work from Smiley (1996), shows that if one has a natural deduction system that contains not just rules for *affirming* sentences but rules for *denying* sentences as well, then it is easy to arrive at a harmonious natural deduction system for classical logic. In such a system, a well-formed formula must be prefaced with a positive or negative force-marker, expressing either affirmation or denial. Thus, the affirmation of a sentence A might be

[5]Notably, Incurvati and Schlöder (2017, 2018, 2023), largely in response to Dickie (2010), extend signed bilateralism to *multi*-lateralism, introducing new signs to express "weak" assertion and rejection, but their systems are still "bilateral" in the sense of the term used here in that they are *at least* bilateral. For other notable proponents or developers of this style of bilateralism, see Humberstone (2000), Francez (2014, 2015), Kurbis (2016, 2019, 2022), Wansing (2016), and Ayhan (2020). There are variations among these bilateralists worth noting. In particular, Wansing (2016), Kurbis (2019), and Ayhan (2020) endorse a somewhat different conception of bilateralism in the development of intuitionistic logic, taking the turnstile itself to be positively or negatively signed, expressing *verification* or *refutation*. Though proponents of this kind of bilateralism take there to be an important conceptual difference between it and standard Rumfitt-style bilateralism (see especially Kurbis (2023) on this point), the signed notation is formally inter-translatable, and so I still take them to fall within this general style.

[6]I leave the notion of harmony informal in this preliminary presentation, as there are many differing conceptions of what, exactly, it amounts to. For discussion, see Steinberger (2011b).

written as $+\langle A\rangle$, and the denial of A can be written as $-\langle A\rangle$.[7] Rather than interpreting existing proof systems with the notions of affirmation and denial, we construct proof systems in which signs expressing affirmation and denial directly figure. For instance, in the natural deduction system proposed by Rumfit, we have the following rules for negation:

$$\frac{-\langle A\rangle}{+\langle\neg A\rangle}\ +\neg_I \qquad\qquad \frac{+\langle A\rangle}{-\langle\neg A\rangle}\ -\neg_I$$

$$\frac{+\langle\neg A\rangle}{-\langle A\rangle}\ +\neg_E \qquad\qquad \frac{-\langle\neg A\rangle}{+\langle A\rangle}\ -\neg_E$$

The sort of reductio that usually figures as the negation introduction rule in Gentzen's natural deduction system, now figures as a distinctively bilateral *structural* rule, called "Smileian Reductio." Where φ and ψ are signed formulas, and starring a formula yields the oppositely signed formula, the principle is the following:

$$\frac{\Gamma, \varphi \vdash \psi \quad \Gamma, \varphi \vdash \psi^*}{\Gamma \vdash \varphi^*}\ \text{Smileian Reductio}$$

This is a structural rule, analogous to Gentzen's other structural rules in that it makes no reference to any specific connectives, but distinctively bilateral in concerning positively and negatively signed formulas. With these negation rules, this added structural rule, and negative conjunction and disjunction rules that exploit the duality of conjunction and disjunction, this style of bilateralism yields a harmonious natural deduction system for classical logic.

So, both bilateralist programs are aimed, in their own ways, at vindicating proof systems for classical logic proposed by Gentzen, the first using bilateralism to interpret Gentzen's multiple conclusion sequent calculus as it stands and the second using bilateralism to tweak Gentzen's natural deduction system to yield a system with desireable formal properties. Despite this striking similarity, there is quite a striking difference between the two programs: they each end up with very different answers to what logic is fundamentally *about*. For

[7] I use angle brackets to be clear that the positive and negative signs always take wide scope over the whole formula. I find it also helps with readability.

the Restall-style bilateralist, logic is fundamentally about *incoherence*, whereas, for the Rumfitt-style bilateralist, logic is fundamentally about *consequence*.[8] This opposition between incoherence and consequence can be made explicit through the use of normative vocabulary developed by Brandom (1994, 2008). The turnstile, on a Restall's interpertation of it, can be understood as expressing the negative normative force of *preclusion of entitlement*: a sequent of the form $\Gamma \vdash \Delta$ says that affirming everything in Γ *precludes one from being entitled* to deny everything in Δ. That is what it is for affirming everything in Γ to be incompatible with denying everything in Δ.[9] For a Rumfitt-style, bilateralist, on the other hand, the turnstile expresses what one might think a turnstile ought to express: a relation of *consequence*. Once again, Brandom's vocabulary is helpful here. A signed sequent of the form $\Gamma \vdash \varphi$ says that taking the stances in Γ (be they affirmations or denials) *commits one* to taking this stance to φ.[10]

It's this notion of committive consequence, I take it, that Rumfit (2008) is speaking of when he speaks of "the force of consequence" which he criticizes Restall's understanding of the turnstile for lacking. In illustrating this notion of force, he considers a hypothetical (or perhaps actual) exchange with a student in one of his seminars:

> What do you mean, you refuse to accept B? You continue to adhere to A, and I've shown you that B follows from A, (80).

Rumfitt takes it that, given the student's acceptance of A and acknowledgment of the fact that *B* follows from *A*, she's *obliged* to accept *B*. That is to say, given

[8]This characterization of Restall-style bilateralism as a position in which logic is *not* about consequence will, of course, come with, for the Restall-style bilateralist will want to say that logical consequence *is* a matter of incoherence, but to say that is to treat "consequence" as a technical term, and, as I am using it here, it is to be understood in terms of the pre-theoretical notion of something's *following from* something else. It's also worth noting that Rosenblatt (2019) puts forward an anti-sequent calculus to extend Restall-style include coherence and incoherence, but the main point of contrast still holds even distinguishing (in)coherence from consequence.

[9]Ripley (2017) explicitly likens the notion of incoherence codified by multiple conclusion sequents, on Restall's bilateral interpretation of them, to Brandom's notion of "material incomaptibility," which Brandom cashes out in terms of this notion of preclusion of entitlement. For Brandom (1994, 2008), a sentence *A* is incompatible with a sentence *B* just in case assertion of (or commitment to) *A precludes entitlement* to *B*.

[10]Incurvati and Schlöder (2017, 2019, 2023) explicitly articulate the rules of their bilateral system as preserving commitment, often with reference to Brandom.

the stances that she has taken, she's *committed* to taking this positive stance towards B. On Restall's understanding of validity, all one can say here is that she is precluded from being entitled to take the negative stance towards B. Of course, Rumfitt acknowledges that is indeed the case, but he thinks it's crucial that we be able to say something stronger here as well. Now, Restall (2013), in defense of his coherence-based account of the logical validity, strikes back against this sort of account in which the turnstile expresses a notion of committive consequence. He says,

> To take an argument to be valid does not mean that when one asserts the premises one should also assert the conclusion (that way lies madness, or at least, making too many assertions). No, to take an argument to be valid involves (at least as a part) the commitment to take the assertion of the premises to stand against the denial of the conclusion, (82).

Clearly, however, this is a cheap shot. The notion of being *committed* to asserting some sentence is indeed a kind of obligation, concerning what one *should* do, rather than what one is *precluded* from doing. Crucially, however, it's a sort of *dispositional* obligation, one which can be *triggered* in various circumstances, rather than a *standing* obligation. Specifically, one is committed to asserting the conclusion of a valid argument whose premises one accept in the sense that one is obligated to assert it *if one is prompted to do so*, as Rumfit prompts his student to accept B in the above quote.

Now, Restall and Rumfitt each have further arguments for their respective conceptions of logical validity and against each other's.[11] Though I think there is something intuitively compelling about Rumfitt's insistence on consequence, properly so-called, it's not clear to me that this dispute between the two conceptions of logical validity can be settled in a logical vacuum. That is, if one considers solely logical vocabulary, it is hard to see any decisive reasons favoring one conception or the other. However, one of the main broader projects motivating the development of bilateralism of both forms is its poten-

[11]For instance, for Restall against Rumfitt, see Restall (2020, 12-14) and Kurbis (2023) for a development of this point, and, for Rumfitt against Restall, see Rumfitt (2015, 51) and Steinberger (2011, 349-353) for a development of this point.

tial application in an inferentialist semantics, not just for logical vocabulary, but for natural language in general.[12] When we try to do this, it becomes clear that the sort of sequent calculi that are readily made sense of by Restall-style bilateralism are better suited to the job of defining the meanings of the logical connectives. It also becomes clear, however, that single conclusion sequents of the sort that figure in Rumfitt-style bilateral systems must play the principal role in formally modeling semantic significance, from an inferentialist perspective. There is reason, then, to want to bring these two styles of bilateralism together so as to be able to reap the benefits of both. Let me explain.

2 Respective Virtues for Inferentialist Semantics

2.1 Virtues of Restall's Bilateralism

If we are providing an inferentialist account of the meanings of the logical connectives in terms of rules in a sequent calculus, and we want to extend our approach to inferentially account for the non-logical meanings of atomic sentences, there's a straightforward way of doing so: we simply include *non-logical axioms*. That is, in addition to having logical axioms of the form $A \vdash A$, we'll also include "material axioms" such as **red** $\vdash$ **colored**.[13] Such material axioms can be taken to be partly constitutive of the meanings of the atomic sentences they relate, and the rules for introducing logically complex sentences, given such axioms, can be understood as partly constitutive of the meanings of the connectives those sentences contain. For instance, it's partly constitutive of the meaning of disjunction that one can move from **red** $\vdash$ **colored** and **blue** $\vdash$ **colored** to **red** $\vee$ **blue** $\vdash$ **colored**. Once one introduces material axioms in this way, however, it's a short step to realizing that one needs to reject some structural rules.

Consider, for instance, that it seems partly constitutive of the meaning of "bird" that one can generally move from the claim that something's a bird to

[12] For expression of this motivation among bilateralists and developments in pursuit of it, see Ripley (2017), Tanter (2021), Francez (2015), and Incurvati and Schlöder (2017, 2019).

[13] I'll take bolded words of this sort as symbols for sentences such as "*a* is red" and "*a* is colored."

the claim that it flies. So, if we want to inferentially account for such sentences as "Bella's a bird" and "Bella flies," we'll want to have **bird** $\vdash$ **flies** as a non-logical material "axiom" in our proof system. However, we won't want to have **bird**, **penguin** $\vdash$ **flies**. Accordingly, we'll have to reject the structural rule of *Weakening*:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, B \vdash A, \Delta}\ \text{Weakening}$$

One important feature of sequent calculi of the sort interpreted by Restall's bilateralism is that the use of structural rules such as Weakening themselves constitute logical steps in the use of the sequent calculus, rather than such rules being built into to structure of the proof system. This makes sequent calculi the natural setting for constructing substructural logics: logical systems that work without the use of such rules. Now, there are different reasons to want a logical system that works without such rules, but, in this inferentialist context, the reason is so that the system is able to accommodate sequents for which they actually fail. In the context of Restall's bilateralism, the failure of Weakening just discussed can be understood in terms of the fact that affirming "Bella's a bird" and denying "Bella flies" constitutes a *curably* "out of bounds" position. That is, there is sort of normative tension between these two acts, but this tension can be *cured* by additional affirmations or denials, such as the additional affirmation of "Bella's a penguin."

Now, Gentzen's LK, the sequent system officially endorsed by both Restall and Ripley, not only requires Weakening in order to function, but the connective rules directly enforce it.[14] For instance, consider one of Gentzen's left conjunction rules:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L_{\wedge_1}$$

This would let us go from the fact that affirming "Bella's a bird" is incompatible with denying "Bella flies" to the fact that affirming "Bella's a bird and she's a penguin" is incompatible with denying "Bella flies," and that is precisely

[14]Well, the sequent system Ripley endorses is LK per se, but LK without the rule of Cut.

what we don't want to say. However, though Gentzen's LK has a problem accommodating defeasible incompatibilities, by tweaking the rules, we can avoid such consequences. In his 1944 dissertation, Oiva Ketonen put forward a classical sequent calculus in which not just Cut, but Weakening as well, is eliminable.[15] Here is Ketonen's classical sequent calculus:[16]

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{Ax}$$

Where Γ, Δ, and $\{A\}$ contain only atomics.

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \mathrm{L}_{\neg} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \mathrm{R}_{\neg}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \mathrm{L}_{\wedge} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ \mathrm{R}_{\wedge}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \mathrm{L}_{\vee} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ \mathrm{R}_{\vee}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ \mathrm{L}_{\to} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ \mathrm{R}_{\to}$$

Note that the axiom schema here is distinct from the more familiar axioms schema of *Reflexivity*: $A \vdash A$. Ketonen's axiom schema generalizes Reflexivity to allow for axioms in which additional formulas have been added in on the left or right. This builds in all of the Weakening one needs for classical logic in the axioms, and so Weakening as a structural rule can be eliminated. Because Weakening is eliminable, this system permits the addition of non-logical material axioms for which Weakening actually fails, and, unlike Gentzen's rules, the rules of this system play nicely with such axioms. For instance, if you look at the conjunction rules, you'll see that we can no longer derive **bird** ∧ **penguin** ⊢ **flies**, from the sequent **bird** ⊢ **flies**. We need the sequent **bird**, **penguin** ⊢ **flies** which we won't include as a material axiom, since it's not a good material inference.

[15] And, moreover (and more technically significantly), Contraction is eliminable as well, but I'll ignore this fact here, as I am, for simplicity, treating sequents as relating sets.

[16] For discussion of the formal properties of this sequent calculus, see Negri and von Plato (2008) and

The move to a Ketonen-style sequent calculus has recently been motivated on these grounds by Brandom (2018), Hlobil (2018), Kaplan (2017, 2022), and Brandom and Hlobil (forthcoming), and this move is one that bilateralists such as Restall and Ripley ought to welcome. Restall (2016) has encouraged proof-theoretic accounts of "concepts beyond the core logical constants," and one substantive step in that direction is providing an account, in terms of defeasible incompatibility relations, of material concepts such as "bird" and "flies." Moreover, Ripley (2017) has explicitly likened the notion of incompatibility codified by multiple conclusion sequents, on Restall's bilateralist interpetation of them, to Brandom's notion of "material incompatibility," and has proposed bilateralism for natural language inferentialist semantics. Because very many of the material incompatibility relations that must be codified to have an adequate account of meaning in natural language are defeasible like the example above, the sequent calculus Ripley should endorse is not Gentzen's, but Ketonen's which can play nicely with a non-monotonic consequence relation. Notably, Ripley (2013) has rejected *Transitivity* as a way of dealing with the semantic paradoxes. However, from the perspective of inferentialist semantics for natural language, a stronger case for the rejection of Transitivity, which does not rely on semantic paradoxes, has to do with the tight connection between Transitivity and Monotonicity. Consider, for instance, that there's a simple way of concocting a failure of Transitivity out of any failure of Monotonicity that involves a general rule with exceptions. To take the same example, we'll presumably want to have **bird** $\vdash$ **flies** and **penguin** $\vdash$ **bird**, but not **penguin** $\vdash$ **flies**. Thus, we have to reject the following rule, which I'll call "Simple Transitivity":

$$\frac{\Gamma \vdash A \quad A \vdash B}{\Gamma \vdash B}\ \text{Simple Transitivity}$$

Moreover, consider the principle of *Cumulative* Transitivity:

$$\frac{\Gamma \vdash A \quad \Gamma, A \vdash B}{\Gamma \vdash B}\ \text{Cumulative Transitivity}$$

This principle is weaker than Simple Transitivity in a non-monotonic context, but even it has clear counter-examples when we consider defeasible reasoning.

For instance, we presumably want $\mathbf{bird} \vdash \mathbf{flies}$ and $\mathbf{bird}, \mathbf{flies} \vdash \neg\mathbf{penguin}$, but not $\mathbf{bird} \vdash \neg\mathbf{penguin}$.[17] Of course, Gentzen's LK is able to accommodate such failures of Transitivity—this is a consequence of Gentzen's Cut-Elimination theorem. However, the Ketonen system shown above is able to accommodate both Monotonicity and Transitivity failures in a unified way. Moreover, these failures make perfect sense on Restall's bilateral interpretation.

Whereas Restall's bilateralism enables us to straightforwardly incorporate non-logical axioms encoding defeasible material inferential relations, it's not at all clear how we can do something similar in the sort of bilateral systems proposed by Rumfitt. The most straightforward way to incorporate material inferences into a natural deduction system would be with primitive inference rules like the following:

$$\frac{+\langle\mathbf{red}\rangle}{+\langle\mathbf{colored}\rangle} \qquad\qquad \frac{+\langle\mathbf{red}\rangle}{-\langle\mathbf{green}\rangle}$$

But, of course, if we add in defeasible inferential rules like the following

$$\frac{+\langle\mathbf{bird}\rangle}{+\langle\mathbf{flies}\rangle} \qquad\qquad \frac{+\langle\mathbf{penguin}\rangle}{+\langle\mathbf{bird}\rangle}$$

we'll be able to link up inferences to illicitly infer **flies** from **penguin**. To block such inferences, it seems that we need a sequent calculus (or something very much like one) in order to keep track of background premises. Still, even transposing a Rumfitt-style bilateral system into sequent notation, it's hard to see how to do without structural rules such as Weakening them when it comes to logically extending a set of material axioms via the operational rules. Consider, for instance, the derivation of $+\langle\mathbf{red} \vee \mathbf{yellow}\rangle \vdash +\langle\neg\mathbf{blue}\rangle$ from $+\langle\mathbf{red}\rangle \vdash -\langle\mathbf{blue}\rangle$ and $+\langle\mathbf{yellow}\rangle \vdash -\langle\mathbf{blue}\rangle$ in Rumfitt's system (transposed into sequent notation):

$$\dfrac{\dfrac{+\langle\mathbf{r}\vee\mathbf{y}\rangle \vdash +\langle\mathbf{r}\vee\mathbf{y}\rangle \quad \dfrac{+\langle\mathbf{r}\rangle \vdash -\langle\mathbf{b}\rangle}{+\langle\mathbf{r}\vee\mathbf{y}\rangle, +\langle\mathbf{r}\rangle \vdash -\langle\mathbf{b}\rangle}\text{Weak.} \quad \dfrac{+\langle\mathbf{y}\rangle \vdash -\langle\mathbf{b}\rangle}{+\langle\mathbf{r}\vee\mathbf{y}\rangle, +\langle\mathbf{y}\rangle \vdash -\langle\mathbf{b}\rangle}\text{Weak.}}{+\langle\mathbf{r}\vee\mathbf{y}\rangle \vdash -\langle\mathbf{b}\rangle}{+\vee_E}}{+\langle\mathbf{r}\vee\mathbf{y}\rangle \vdash +\langle\neg\mathbf{b}\rangle}{+\neg_I}$$

[17] See Simonelli (2022) for an extended discussion of these examples, as they arise in the context of natural language indicative conditionals.

Moreover, the operational rules of Rumfit's system enforce the structural rule of Weakening in just the way those of Gentzen's LK do. Consider, for instance, the negative conjunction rules proposed by Rumfitt, exploiting the duality of conjunction and disjunction:

$$\frac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash -\langle A \wedge B \rangle} -\wedge_{\mathrm{I}_1} \qquad\qquad \frac{\Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \wedge B \rangle} -\wedge_{\mathrm{I}_2}$$

In this context, we can see that there is clearly a problem with these negative conjunction rules. For instance, let Γ be an affirmation of 'Sadie lays eggs," A be "Sadie's a mammal," and B be "Sadie's a platypus." Affirming "Sadie lays eggs," in general, commits one to denying "Sadie's a mammal," but it doesn't commit one to denying "Sadie's a mammal and she's a platypus." On the contrary, saying "Sadie's a mammal and she's a platypus" is perfectly compatible with affirming "Sadie lays eggs."[18]

Now, perhaps the operational rules can be modified to avoid this problem.[19] However, an even more serious problem for Rumfitt-style systems in this context is their reliance on bilateral structural rules. Consider again the principle of Smileian Reductio:[20]

$$\frac{\Gamma, \varphi \vdash \psi \quad \Gamma, \varphi \vdash \psi^*}{\Gamma \vdash \varphi^*} \text{ Smileian Reductio}$$

Smiley (1996) says that, in bilateral systems, this is "The one principle that is always necessary for completeness, either as a primitive or a derived rule," (5). If that's true, it's bad news for the Rumfitt-style bilateralist if they want to incorporate defeasible inferential relations into their system. To see this, let Γ be the set consisting in an affirmation of "Sadie's a mammal" and an affirmation of "Sadie lays eggs," and let φ be the denial of "The moon is made of cheese."

[18]It's worth emphasizing that, even just in the context of providing an account of the meanings of the logical connectives (bracketing the question of providing an account of the meanings of such expressions as "mammal" and "platypus"), this is a serious problem, since the sentences with which we use logical connectives such as conjunction include such sentences as these sentences about animals.

[19]As we'll soon see, indeed they can be.

[20]It's worth noting that Incurvati and Schlöder (2019) endorse a variation of this principle involving (strong) affirmation and *weak* denial, but the same example applies to the modified principle they endorse as well.

Given that affirming "Sadie's a mammal" commits one to denying "Sadie lays eggs," it seems clear that affirming "Sadie's a mammal" along with affirming "Sadie lays eggs" and denying "The moon is made of cheese" still commits one to denying "Sadie lays eggs," and, just as well, since this set of stances contains an affirmation of "Sadie lays eggs" it, of course, commits one to affirming "Sadie lays eggs." Given Smileian Reductio, then, it follows that affirming "Sadie's a mammal" and affirming "Sadie lays eggs" commits one to affirming "The moon is made of cheese." Such an explosion of commitments in a case of someone's making two affirmations that are only *defeasibly* incompatible, however, seems problematic. After all, someone who affirms both "Sadie's a mammal" and affirms "Sadie lays eggs" may further affirm "Sadie's a platypus," thereby curing the incoherence of their set of affirmations. Smileian Reductio, which is thought to be ineliminable from Rumfitt-style systems, precludes us from being able to say such a thing.

2.2 Virtues of Rumfitt-Style Systems

The above difficulties nonwithstanding, there is substantial reason to want a Rufmitt-style system for the purpose of inferentialist semantics. By far, the most philosophically well-developed and well-motivated framework for thinking about meaning in terms of inferential rules has been provided by Robert Brandom (1994, 2000). Accordingly, it seems reasonable to want to contextualize any formal inferentialist semantics in a Brandomian framework. Now, Brandom's inferentialism is a *normative pragmatic* inferentialism. Inferential rules are understood, in the first instance, as principles for keeping discursive score, which determine what the utterance of a sentence *does*, normatively speaking, in a discursive practice in which it might be uttered. That is, the inferential or discursive role of a sentence is understood in terms of how the utterance of it changes the "social deontic score," the normative statuses that have been assigned to the various participants of the discursive practice. In Brandom's words:

> The significance of an assertion of *p* can be thought of as a mapping that associates with one social deontic score—characterizing

the stage before that speech act is performed, according to some scorekeeper—the set of scores for the conversational stage that results from the assertion, according to the same scorekeeper (1994, 190).

As Bernhard Nickel (2013) has already made explicit, spelling this idea out formally, we're going to have a *dynamic* semantics, a semantic theory in which the meaning of a sentence is understood in terms of its potential to update a context.[21] The context change potential of a sentence is a function mapping each context in which it might be uttered to the one that would result upon its being uttered. In a standard dynamic semantic theory, which draws its initial inspiration from Stalnaker's (1978) pragmatics for assertion, contexts are taken to be sets of possible worlds.[22] In a Brandomian dynamic semantics, however, contexts are taken to be characterizations of the social deontic score.

Nickel's simple proposal for formalizing Brandom's normative pragmatic inferentialism as a kind of dynamic semantics is to take context to be sets of sentences, those to which discursive participants are committed. However, it's quite clear from the logical problems that Nickel immediately encounters on his proposal, that we need somewhat fancier contexts. There are different ways to go here, but I'll propose two upgrades to Nickel's simple proposal. The first upgrade is going bilateral, distinguishing between *affirmations* and *denials* of sentences. The second upgrade is to distinguish between the affirmations and denials that one has actually *made* and those to which one is *committed*. So, the contexts will be "scorecards" which specify the affirmations and denials a speaker has made and the affirmations and denials to which that speaker is committed. Sequents, then, are interpreted as "scorekeeping principles," which determine how a scorecard is updated when a speaker makes some affirmation or denial; we can read a bilateral sequent of the form $\Gamma \vdash \varphi$ as saying that, whenever someone has made moves in Γ, they are committed to making the move φ, be it an affirmation or denial. The proof rules of a

[21]This, at least, is the most austere way of carrying out Brandom's dictum that "semantics must answer to pragmatics." Of course, assigning semantic values in this way is not compulsory, but, even if one opts to define semantic values in some other way, a pragmatic framework fo this sort is still necessary, and so the following points still hold.

[22]See Veltman (1996) for a classic theory of this sort

bilateral sequent calculus, then, can be understood as rules for the expansion of scorekeeping principles, which tell us how, given a set of scorekeeping principles relating stances towards atomic sentences, we can arrive at a set of scorekeeping principles relating stances towards sentences of arbitrary logical complexity which enable us to define updates for logically complex sentences.

For concreteness, let me briefly lay out what a formal framework along these lines might look like. For our purposes, we can take the "scorekeeping game" to involve just two participants, a designated scorekeeper, with a certain set of scorekeeping principles, and a designated move-maker, who makes the moves. We now define:

> **Scorecards:** A *scorecard* σ is a pair of sets of signed formulas of the form $\langle \sigma_m, \sigma_c \rangle$, where σ_m is the set of moves that a player has *made* and σ_c is the set of moves to which a player is *committed*.

> **Scorekeeping Principles:** A *scorekeeping principle* is a bilateral sequent of the form $\Gamma \vdash \varphi$, where Γ is a set of signed formulas and φ is a single signed formula.

The intuitive idea is that $\Gamma \vdash \varphi$ says that, whenever a player has *made* the moves in Γ, they're *committed* to φ. Officially, we might define the operation of applying a set of scorekeeping principles to a set of moves to determine the commitments of someone who has made those moves as follows (we will revise this definition shortly):

> **Application of Scorekeeping Principles (Version 1):** The result of applying a set of scorekeeping principles π to a set of moves M, which we denote $\pi(\mathrm{M})$ is the smallest superset of M such that, for every scorekeeping principle of the form $\Gamma \vdash \varphi \in \pi$ if $\Gamma \subseteq \mathrm{M}$ then $\varphi \in \pi(\mathrm{M})$.

It is now straightforward to define the update on a scorecard effected by the affirmation or denial of a sentence A. The set of moves made, in the updated scorecard, will simply be the set of moves made in the original scorecard plus the additional affirmation/denial of A, and the set of commitments will be the result of applying the scorekeeping principles to this updated set of moves made. That is:

Updates: The result of updating a scorecard σ with a move φ, $\sigma[\varphi]$, is $\langle \sigma_m \cup \{\varphi\}, \pi(\sigma_m \cup \{\varphi\})\rangle$

This lets us define, for a scorekeeper with a certain set of scorekeeping principles, the set of scorecards that are *possible*, relative to that player:

Possible scorecards: For a scorekeeper with any set of scorekeeping principles, π, the set of possible scorecards, relative to that scorekeeper, is recursively defined as follows:

1. $\langle \varnothing, \pi(\varnothing)\rangle \in \Sigma$
2. For any $\sigma \in \Sigma$, any move φ and $\sigma[\varphi] \in \Sigma$

This enables us to define semantic values as update functions of just the sort suggested by Brandom. Assuming that assertorically uttering A has the same discursive significance as affirming A, the semantic value of a sentence A is a function that maps any scorecard one might have to the scorecard that would result upon the move-maker's assertoric utterance of A. That is:

Semantic Values:

$$\begin{aligned} &[\![A]\!] = f : \Sigma \to \Sigma \\ &\quad f(\sigma) = \sigma[+\langle A\rangle] \end{aligned}$$

This naturally yields an interpretation of single conclusion bilateral sequents as expressing principles of *committive consequence*: a sequent of the form $\Gamma \vdash \varphi$ says that someone who has made the moves in Γ is committed to φ. In general, such sequents tell us how to update our scorecard when someone has made some move. A sequent calculus can then be understood as providing a set of *rules for expending scorekeeping principles,* enabling us to recursively define semantic values for an infinite number of logically complex sentences, given a finite set of scorekeeping principles that determine the semantic values of a finite set of atomic sentences.

The formal details of this semantic proposal are not particularly important for the moment. The main point of note right now is that only single conclusion sequents can be put to use in an update semantics to define semantic values in this way. Consider, for instance, the following sequent, derivable in any classical Rumfitt-style system:

$$+\langle A \vee B\rangle, -\langle A\rangle \vdash +\langle B\rangle$$

Interpreted in this way, this says that affirming $A \vee B$ and denying A commits one to affirming B. In this way, a Rumfitt-sytle system, through which we can derive such sequents, tells us how to update our scorecard when someone affirms a disjunction: if they've also denied one of the disjuncts, they're committed to affirming the other. By contrast, consider the sort of sequent we derive in a multiple-conclusion sequent calculus:

$$A \vee B \vdash A, B$$

On Restall's interpretation, this says that affirming $A \vee B$, denying A, and denying B is out of bounds. That's true, of course, but it doesn't tell us how to attribute commitments to someone who has affirmed a disjunction. In general, a multiple conclusion sequent of the form $\Gamma \vdash \Delta$ simply doesn't tell you what to do when someone makes all of the moves in Γ. It tells you that you *can't* score them as *entitled* to all of the moves in Δ, but that doesn't amount to telling you what you *should* score them as *committed* to. Such sequents can function to *constrain* scorekeeping practices, but they can't function to *dictate* scorekeeping practices, not without being transformed, in some way, to single conclusion sequents.[23] So, beyond any purely philosophical reasons articulated by Rumfitt for preferring a single conclusion system in which the turnstile expresses a relation with "the force of consequence," there is a concrete technical reason to prefer such a system. Insofar as we want to be able to define semantic values in the way, we need single conclusion sequents as the "scorekeeping principles" that determine the semantic significance of the utterance of sentences. Rumfitt-style signed sequents are able to serve this function, but multiple conclusion sequents are not.

[23] It's worth noting that, in a similar Brandomian context, multiple conclusion sequents have been interpreted as constraints on scorecards by Kukla, Lance, and Restall (2009, 225). I take the framework proposed here to be an advance on that framework for these reasons (among others).

3 A Bridge Between Bilateralisms

So, in the context of providing an inferentialist semantics, both forms of bilateralisms have benefits, but both have serious limitations. There is reason, then, to wonder if we can bring both forms of bilateralism together so as to get the best of both worlds. We can, and I'll now show how.

Let us start with Restall's bilateralism. Recall, for Restall, a sequent of the form $\Gamma \vdash \Delta$ says that the position consisting in affirming everything in Γ and denying everything in Δ is incoherent. Now, in a unilateral sequent calculus, interpreted in a standard fashion, a sequent of the form $\Gamma \vdash$ is understood as codifying the fact that the set of sentences in Γ are jointly incoherent. It is reasonable to think, then, that in a bilateral sequent calculus, such a sequent can express the same thing. That is, where Γ' is a set of *signed* sentences, $\Gamma' \vdash$ says that the set of stances in Γ', be they affirmations or denials, are jointly incoherent. This suggests the straightforward translation of Restall's bilateralism into the signed notation proposed by Rumfitt; unsigned sequents of the form $\Gamma \vdash \Delta$ simply get mapped to sequents of the form $\Gamma' \vdash$, where Γ' is a set of signed formulas. In particular, to translate an unsigned multiple conclusion sequent of the form $\Gamma \vdash \Delta$, on Restall's interpretation, to a signed sequent of the form $\Gamma' \vdash$ let $\Gamma' = \{+\langle A \rangle \mid A \in \Gamma\} \cup \{-\langle A \rangle \mid A \in \Delta\}$. Conversely, to translate a signed sequent of the form $\Gamma' \vdash$ to unsigned multiple conclusion sequent of the form $\Gamma \vdash \Delta$ let $\Gamma = \{A \mid +\langle A \rangle \in \Gamma'\}$ and $\Delta = \{A \mid -\langle A \rangle \in \Gamma'\}$.

This is a faithful one-to-one translation, and so whole sequent systems can be translated in this manner. For instance, deploying this simple translation schema, the negation rules of a multiple conclusion classical sequent calculus come out, in explicitly bilateral notation, as follows (where Γ now is a set of signed sentences):

$$\frac{\Gamma, -\langle A \rangle \vdash}{\Gamma, +\langle \neg A \rangle \vdash}\ +_{\neg} \qquad\qquad \frac{\Gamma, +\langle A \rangle \vdash}{\Gamma, -\langle \neg A \rangle \vdash}\ -_{\neg}$$

The positive rule, which translates the left negation rule of the classical sequent calculus, says that if the position consisting in all of the affirmations and denials in Γ along with the denial of A is out of bounds, then the position consisting in all of the affirmations and denials in Γ along with the affirmation of $\neg A$ is out of bounds. The negative rule, which corresponds to the right rule of the classical sequent calculus, says that if the position consisting in all of the affirmations and denials in Γ along with the affirmation of A is out of bounds, then the position consisting in all of the affirmations and denials in Γ along with the denial of $\neg A$ is out of bounds. In a similar way, the rest of a multiple conclusion sequent calculus such as Ketonen's can be translated in this way. These rules with signed formulas now *show*, explicitly in the bilateral notation itself, exactly what the more familiar sequent rules *say*, on a bilateralist interpretation of them.

Thus far, I have simply *stated* that a sequent of the form $\Gamma \vdash$ expresses the incoherence set of stances Γ. This, of course, should be simply stated, but, rather, should be *codified* in the rules of sequent system itself. Now, in a (classical) unilateral system, the negation rules can be understood as formally codifying the fact that a sequent of the form $\Gamma \vdash$ encodes the joint incoherence of the sentences in Γ.[24] Thus, for instance, the sequent:

$$\mathbf{red}, \mathbf{green} \vdash$$

can be understood as encoding the *incoherence* of the set consisting in these sentences, since, with the negation rules, we can derive from this sequent:

$$\mathbf{red} \vdash \neg\mathbf{green}$$

and

$$\mathbf{green} \vdash \neg\mathbf{red}$$

In general, a sequent of the form $\Gamma \vdash$ can be understood as formally encoding the fact that the set of sentences Γ is incoherent, as borne out by the fact that, for all Γ', where $\Gamma' = \Gamma \setminus \{A\}$ with $A \in \Gamma$, $\Gamma' \vdash \neg A$. In a bilateral system, we can get the same behavior at the structural level by way of following pair of rules (where Γ is now a set of signed sentences):

[24] One might alternately think that the fact that a sequent of the form $\Gamma \vdash$ encodes the incoherence of Γ is formally codified by the fact that, if you have such a sequent, you can, with Monotonicity, conclude that $\Gamma \vdash A$, for any sentence A. However, we rejected Monotonicity here, and so the codification in terms of the negation rules is much more apt.

$$\frac{\Gamma, \varphi \vdash}{\Gamma \vdash \varphi^*}\ \text{Out} \qquad\qquad \frac{\Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash}\ \text{In}$$

The Out rule can be understood as saying that, if the position consisting in all of the stances in Γ along with stance φ is incoherent, then Γ commits one to taking the opposite stance φ^*, whereas the In rule can be understood as saying that, if Γ commits one to taking the stance φ, then the position consisting in Γ along with the opposite stance φ^* is incoherent. With these rules in view, consider the following sequent:

$$+\langle\textbf{red}\rangle, +\langle\textbf{green}\rangle \vdash$$

This says that the position consisting in affirming "*a* is red" and affirming "*a* is green" is incoherent. The incoherence of the position consisting in both of these affirmations can be understood in terms of the fact that affirming "*a* is red" commits one to denying "*a* is green" and affirming "*a* is green" commits one to denying "*a* is red." The relation between all of these incoherence and incompatibility facts is codified by In and Out, as, given these rules, this sequent is equivalent to this one:

$$+\langle\textbf{red}\rangle \vdash -\langle\textbf{green}\rangle$$

and this one:

$$+\langle\textbf{green}\rangle \vdash -\langle\textbf{red}\rangle$$

Whereas the sequent with both affirmations on the left can be understood as encoding an *incoherence property* of that set of sentences, these sequents, with an affirmation on the left and a denial on the right can be understood as encoding an *incompatibility relation* between the sentences.

Given our explicitly bilateral translation of Restall's bilateralism, In and Out, in effect, constitute a *bridge between bilateralisms*, enabling us to move from (faithfully translated) mutliple conclusion-sequents, encoding *incoherence*, to Rumfitt-style sequents, encoding *consequence*, and vice versa. Of course, the actual acceptance of such rules is sure to be controversial among many logicians, not least of whom will be Restall and Ripley, who want to resist a committive notion of logical consequence.[25] Nevertheless, I take it that all parties should welcome a formal framework of this sort in which the moves that would bridge bilateralism are formally represented.[26] Regardless of whether one accepts In and Out as sound bilateral structural rules, one can systematically investigate various different possibilities for bilateralism of both sorts, with different bilateral structural rules and different operational rules, logically mapping the philosophical landscape. There is much to be explored if one considers bilateral systems that don't impose In and Out. However, here, I want to consider one system that makes use of these rules in order to meet the desiderata for an inferentialist logic articulated in the previous section.

4 The Best of Both Worlds

We noted above the benefits of the Kentonen-style rules when it came to providing a proof-theoretic account of the meanings of the logical connectives when material axioms are added into the language. Reformulated in this bilateral notation, Ketonen's sequent calculus is the following:

$$\frac{}{\Gamma, \varphi, \varphi^* \vdash}\ \text{Ax}$$

Where Γ and $\{\varphi\}$ contain only signed atomics.

$$\frac{\Gamma, -\langle A \rangle \vdash}{\Gamma, +\langle \neg A \rangle \vdash}\ +_\neg \qquad \frac{\Gamma, +\langle A \rangle \vdash}{\Gamma, -\langle \neg A \rangle \vdash}\ -_\neg$$

$$\frac{\Gamma, +\langle A \rangle, +\langle B \rangle \vdash}{\Gamma, +\langle A \wedge B \rangle \vdash}\ +_\wedge \qquad \frac{\Gamma, -\langle A \rangle \vdash \quad \Gamma, -\langle B \rangle \vdash}{\Gamma, -\langle A \wedge B \rangle \vdash}\ -_\wedge$$

[25] I mentioned Restall's resistance earlier, but of particular note here is Ripley's (2013) non-transitive approach to the liar paradox that crucially hangs on the bilateral reading of the turnstile, and becomes much less intuitive if one applies Out to move from $-\langle \lambda \rangle \vdash$, a sequent that can be derived in Ripley's system which says that one can't coherently deny liar sentence, to $\vdash +\langle \lambda \rangle$, which says that one is committed to affirming it.

[26] In this way, the multiple steps involved in this bridge between bilateralisms—which first translates multiple conclusion sequents exactly in a way that simply makes Restall's bilateral interpretation explicit, and then imposes bilateral structural rules to transform them into Rumfitt-style sequents—constitutes a decisive advantage over existing proposals for translating between multiple conclusion sequents into Rumfitt-style bilateral sequents, such as Hjortland's (2014, 444), which simply translates $\Gamma \vdash \Delta$ as $+\langle \gamma_1 \rangle, +\langle \gamma_2 \rangle \ldots +\langle \gamma_n \rangle, -\langle \delta_2 \rangle \ldots -\langle \delta_n \rangle \vdash +\langle \delta_1 \rangle$.

$$\frac{\Gamma, +\langle A\rangle \vdash \quad \Gamma, +\langle B\rangle \vdash}{\Gamma, +\langle A \vee B\rangle \vdash}\ +_{\vee} \qquad \frac{\Gamma, -\langle A\rangle, -\langle B\rangle \vdash}{\Gamma, -\langle A \vee B\rangle \vdash}\ -_{\vee}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \quad \Gamma, +\langle B\rangle \vdash}{\Gamma, +\langle A \to B\rangle \vdash}\ +_{\to} \qquad \frac{\Gamma, +\langle A\rangle, -\langle B\rangle \vdash}{\Gamma, -\langle A \to B\rangle \vdash}\ -_{\to}$$

Extending a set of material-incompatibility-encoding base sequents such as $+\langle \textbf{red}\rangle, +\langle \textbf{green}\rangle \vdash$, $+\langle \textbf{bird}\rangle, -\langle \textbf{flies}\rangle \vdash$, and so on by way of this calculus, and then applying Out to all of the resultant sequents gives us principles of committive consequence needed to figure in a scorekeeping framework of the sort laid out above. However, while this approach works fine technically, I take it that many will find these rules, which relate *incoherences* to be less intuitive than rules that relate *consequence* of the sort that figure in Rumfitt's natural deduction system. Insofar as we are thinking of the meanings of the logical connectives in terms of the rules governing their use in such a proof system, this is not a completely negligible concern. However, it is easy to address.

Rather than applying Out to get principles of committive consequence at a second stage, we can use Out to rewrite the sequent calculus itself so as to turn it from a calculus of *incoherence* to a calculus of *consequence*. Rewriting each of the above rules with the use of Out, we get the following sequent calculus:

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \text{CO}$$

Where Γ and $\{\varphi\}$ contain only signed atomics.

$$\frac{\Gamma \vdash -\langle A\rangle}{\Gamma \vdash +\langle \neg A\rangle}\ +_{\neg} \qquad \frac{\Gamma \vdash +\langle A\rangle}{\Gamma \vdash -\langle \neg A\rangle}\ -_{\neg}$$

$$\frac{\Gamma \vdash +\langle A\rangle \quad \Gamma \vdash +\langle B\rangle}{\Gamma \vdash +\langle A \wedge B\rangle}\ +_{\wedge} \qquad \frac{\Gamma, +\langle A\rangle \vdash -\langle B\rangle}{\Gamma \vdash -\langle A \wedge B\rangle}\ -_{\wedge}$$

$$\frac{\Gamma, -\langle A\rangle \vdash +\langle B\rangle}{\Gamma \vdash +\langle A \vee B\rangle}\ +_{\vee} \qquad \frac{\Gamma \vdash -\langle A\rangle \quad \Gamma \vdash -\langle B\rangle}{\Gamma \vdash -\langle A \vee B\rangle}\ -_{\vee}$$

$$\frac{\Gamma, +\langle A\rangle \vdash +\langle B\rangle}{\Gamma \vdash +\langle A \to B\rangle}\ +_{\to} \qquad \frac{\Gamma \vdash +\langle A\rangle \quad \Gamma \vdash -\langle B\rangle}{\Gamma \vdash -\langle A \to B\rangle}\ -_{\to}$$

Clearly, given In and Out, this system is equivalent to the previous one. However, we actually don't need In and Out in their full generality in order for this system to be complete. Rather, we need only the following rule, derived from them, which Smiley (1996) calls Reversal:

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma, \psi^{*} \vdash \varphi *}\ \text{Reversal}$$

In standard Rumfitt-style systems, Reversal is treated as a derived rule from Smiliean Reductio, and so little independent attention is given to it. However, Reversal alone, without Smiliean Reductio, is sufficient for the completeness of the above sequent calculus. Unlike the previous system, which directly translates Ketonen's sequent calculus on Restall's interpretation of it, we now we have a system that can properly be conceived of as providing *introduction rules* for positively and negatively signed formulas, specifying the conditions under which one is *committed* to affirming or denying a logically complex sentence. Indeed, the negation rules, the conditional rules, the positive conjunction rule, and the negative disjunction rule are the familiar introduction rules from Rumfitt's system. The less familiar ones are the negative conjunction and positive disjunction rules, though these have recently been proposed, for independent reasons, by del Valle Inclan and Schlöder (2023) in the context of a Rumfitt-style natural deduction system. The negative conjunction rule says that if a set of stances Γ along with an affirmation of A commits one to denying B, then Γ commits one to denying $A \wedge B$. Note that, given Reversal, if Γ along with an affirmation of A commits one to denying B, then, just as well, Γ along with an affirmation of B commits one to denying A. So, essentially, this rule for conjunction says that you're committed to denying a conjunction just in case affirming one of the conjuncts commits you to denying the other. Dually, the positive disjunction rule says that you're committed to affirming a disjunction just in case denying one of the disjuncts commits you to affirming the other.

I'll call the sequent calculus constituted by these operational rules, the structural rule of Reversal, and the axiom schema of Containment (now stated with signed sentences) "BK" for "Bilateral Ketonen." It is easy to show that it is equivalent to Kenton's multiple conclusion sequent calculus in that every proof in this system corresponds to a unique proof in Ketonen's system and every proof in Ketonen's system corresponds to an equivalence class of proofs in this system under Reversal. From a purely intuitive standpoint, I take this set of rules to be at least as good of a candidate for a proof-theoretic specification of the meanings of the logical connectives as the rules in the systems proposed by Smiley or Rumfitt. Technically, however, there are several benefits of this system. Like Ketonen's sequent calculus, this system requires neither Monotonicity nor Transitivity in order to function. Moreover, it requires none of the usual bilateral structural rules appealed to in Rumfitt-style systems, such as Smiliean Reductio.[27] The only bilateral structural rule is Reversal, which we have given an intuitive justification above and which causes none of the problems of rules like Smiliean Reductio. So, we get all the benefits of using Ketonen's multiple-conclusion sequent calculus articulated above, and, if we allow applications of Reversal where $\{\varphi\}$ or $\{\psi\}$ are null (i.e. applications of In and Out), we retain Restall's interpretation of it if we consider only the solely left-sided fragment of the consequence relation. However, we now have a turnstile expressing a relation of *consequence*, properly so-called, between sets of affirmations and denials and single affirmations or denials. Accordingly, we can think of the sequents with formulas on the right-hand side as "scorekeeping principles" of the sort articulated in Section 2.2 above: rules for attributing commitments to affirmations or denials to speakers on the basis of the affirmations and denials they've made. Taking any set of scorekeeping principles π to be closed under the rules of this sequent calculus, this calculus can be understood as a way of extending a finite set of material scorekeeping principles relating affirmations and denials of atomic sentences to an infinite set relating affirmations and denials to logically complex sentences, thus determining the update for the utterance of any logically complex sentence. As promised, this gives us the best of both worlds.

Consider first how we can use this system to logically extend material axioms without the use of structural rules like Weakening or Smiliean Reductio. Consider again the example from Section 2.1 of derriving $+\langle \textbf{red} \vee \textbf{yellow} \rangle \vdash +\langle \neg \textbf{blue} \rangle$ from $+\langle \textbf{red} \rangle \vdash -\langle \textbf{blue} \rangle$ and $+\langle \textbf{yellow} \rangle \vdash -\langle \textbf{blue} \rangle$. Recall, this required the use of Weakening in Rumfitt's system. In this system, no Weakening or Reductio is required:

[27]It's not hard to see why: Given our translation, Smiliean Reductio in this bilateral sequent calculus closely corresponds to the structural rule of *Cut* in a multiple conclusion sequent calculus. In fact, as I show elsewhere (Simonelli M.S.), one can prove that Smiliean Reductio is admissible in the classical fragment of this sequent calculus in a way that is directly analogous to the standard proof of Cut-Elimination.

$$
\dfrac{\dfrac{\dfrac{\dfrac{+\langle \mathbf{r} \rangle \vdash -\langle \mathbf{b} \rangle}{+\langle \mathbf{b} \rangle \vdash -\langle \mathbf{r} \rangle}\,\mathrm{RV} \qquad \dfrac{+\langle \mathbf{y} \rangle \vdash -\langle \mathbf{b} \rangle}{+\langle \mathbf{b} \rangle \vdash -\langle \mathbf{y} \rangle}\,\mathrm{RV}}{+\langle \mathbf{b} \rangle \vdash -\langle \mathbf{r} \vee \mathbf{y} \rangle}\,-_{\vee}}{+\langle \mathbf{r} \vee \mathbf{y} \rangle \vdash -\langle \mathbf{b} \rangle}\,\mathrm{RV}}{+\langle \mathbf{r} \vee \mathbf{y} \rangle \vdash +\langle \neg \mathbf{b} \rangle}\,+_{\neg}
$$

Moreover, consider how the negative conjunction rule avoids the problem faced by the standard pair of negative conjunction rules. Once again, the standard rules are as follows:

$$
\frac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\,-\wedge_{\mathrm{I}_1} \qquad\qquad \frac{\Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\,-\wedge_{\mathrm{I}_2}
$$

Recall, the problem is that, though affirming to "Sadie's a mammal" commits one to denying "Sadie lays eggs," affirming "Sadie's a mammal" doesn't commit one to denying "Sadie's a platypus and she lays eggs." This problem is avoided with our single negative conjunction rule:

$$
\frac{\Gamma, +\langle \varphi \rangle \vdash -\langle \psi \rangle}{\Gamma \vdash -\langle \varphi \wedge \psi \rangle}
$$

This rule precludes one from being able to derive the problematic sequent $+\langle \textbf{mammal} \rangle \vdash -\langle \textbf{platypus} \wedge \textbf{lays eggs} \rangle$, since, though one *will* have the material axiom $+\langle \textbf{mammal} \rangle \vdash -\langle \textbf{lays eggs} \rangle$, one *won't* have the material axiom $+\langle \textbf{mammal} \rangle, +\langle \textbf{platypus} \rangle \vdash -\langle \textbf{lays eggs} \rangle$, which is what one needs in order to apply this negative conjunction rule and derrive the problematic sequent.

Given that we now have a non-monotonic single conclusion sequent calculus, let us modify the above definition of the operation of applying a set of scorekeeping principles to a set of moves in such a way as to accommodate failures of monotonicity. This is easy to do. We simply take it that, for some scorekeeping principle of the form $\Gamma \vdash \varphi$, the attribution of commitment to φ of a player who has made the moves Γ can be *defeated* if that player has also made the moves Δ, and one does not have a principle of the form $\Delta, \Gamma \vdash \varphi$, and, moreover, that such defeat can itself be defeated. Officially:

> **Application of Scorekeeping Principles (Version 2):** The result of applying a set of scorekeeping principles π to a set of moves M,

which we denote $\pi(\text{M})$ is the smallest superset of M such that, for every scorekeeping principle of the form $\Gamma \vdash \varphi \in \pi$ if $\Gamma \subseteq \text{M}$ then either $\varphi \in \pi(\text{M})$ or the following conjunctive condition, the *defeat condition* holds (where Δ and Θ are non-empty):

1. There is some set of moves $\Delta \subseteq \text{M}$ such that $\Delta, \Gamma \vdash \varphi \notin \pi$
2. There is no set of moves $\Theta \subseteq \text{M}$ such that $\Theta, \Delta, \Gamma \vdash \varphi \in \pi$

This definition of scorekeeping principle application lets us define two notions of validity, relative to a set of scorekeeping principles π:

Strict Validity: An inference of the form $\Gamma : \varphi$ is *strictly* valid, $\Gamma \vDash_s \varphi$, just in case, for every set of moves M, $\varphi \in \pi(\text{M} \cup \Gamma)$.

General Validity: An inference of the form $\Gamma : \varphi$ is *generally* valid, $\Gamma \vDash_g \varphi$, just in case $\varphi \in \pi(\Gamma)$.

In the context of the definitions of possible scorecards and updates in Section 2.2, these definitions amount to saying that $\Gamma : \varphi$ is *strictly* valid just in case someone who is scored as having made the moves in Γ, no matter which other moves they are scored as having made, is scored as committed to φ, whereas $\Gamma : \varphi$ is *generally* valid just in case someone who is scored as having made the moves in Γ, and is not previously scored as having made any moves, is scored as committed to φ. Thus, $+\langle\textbf{red}\rangle \vdash +\langle\textbf{colored}\rangle$ is strictly valid, since, no matter what other moves one has made, one who affirms "*a* is red" is committed to affirming "*a* is colored," whereas $+\langle\textbf{bird}\rangle \vdash +\langle\textbf{flies}\rangle$ is only generally valid, since someone who affirms "Bella's a bird," without having antecedently been scored as having made any moves, is committed to affirming "Bella flies," but someone who affirms "Bella's a bird" and "Bella's a penguin" is not. Taking any set of scorekeeping principles π to be the result of closing a base set of scorekeeping principles (relating affirmations and denials to atomic sentences) under rules of BK, it is easy to show that, given these two definitions of validity, (1) any classically valid inference is strictly valid, (2) Monotonicity holds with respect to strict validities but can fail with respect to general validities, and (3) Cumulative Transitivity can fail with respect to both general and strict validities. So, the scorekeeping framework is capable of harmonizing with the sequent system in just the way we want.[28]

The conception of consequence afforded by this scorekeeping framework is importantly different from standard conceptions. Standard conceptions, following Tarski (1930), take the operation of extracting consequences from a set of sentences to be a *closure* operation. By contrast, the operation of applying a set of scorekeeping principles to a set of moves defined here meets only one of the three conditions for a closure operation. It is *extensive*, but it's neither *monotonic* nor *idempotent*. That is, $M \subseteq \pi(M)$, but it's not necessarily the case that if $M \subseteq M'$, then $\pi(M) \subseteq \pi(M')$ nor is it necessarily the case that $\pi(\pi(M)) = \pi(M)$. The non-monotonicity—provided by the defeat condition—is of course what enables us to accommodate failures of Weakening. For instance, in general, if someone affirms "Sadie's a mammal," they'll be taken to be committed to denying "Sadie lays eggs," but if they affirm "Sadie's a mammal" along with "Sadie's a platypus," they won't be taken to be committed to denying "Sadie lays eggs," but if they affirm "Sadie's a mammal," "Sadie's a platypus," and "Sadie's a male," they will be, and so on. The non-idempotency is what enables us to accommodate failures of Cumulatively Transitive, enabling us to say, for instance, that affirming "Bella's a bird" commits one to affirming "Bella flies," and affirming "Bella's a bird" along with affirming "Bella flies" commits one to denying "Bella's a penguin," but affirming "Bella's a bird" does not, by itself, commit one to denying "Bella's a penguin." This is perhaps the most distinctive feature of this framework.

As mentioned above, Ripley (2013) has notably rejected Transitivity in re-

[28] One might wonder about (3), given that the failures of CT given here have all involved defeasible inferential relations. However, I take it that there is good reason to want a framework that can handle CT failures even for strict validities. Consider a case in which Lois Lane says "Superman flies" and "Clarke Kent doesn't fly." Plausibly, given her first assertion, we'll score here as committed to affirming "Clarke Kent flies," and, given her second assertion, we're going to score her as committed to denying "Clarke Kent flies." Both of these commitment attributions are indefeasible, and so she'll be indefeasibly committed to affirming "Clarke Kent flies and Clarke Kent doesn't fly." Given that our system strictly validates every classical validity, affirming "Clarke Kent flies and Clarke Kent doesn't fly" indefeasibly commits one to affirming "The moon is made of cheese." But, we presumably don't want to score Lois Lane—who's merely misinformed and not patently illogical—as committed to "The moon is made of cheese." So, we have to good reason to deny Cumulative Transitivity even for strict validities.

sponding to the liar paradox.[29] Conceptually, however, the sort of failure of transitivity at play here—a failure of transitivity of committive consequence—is quite distinct from the sort of failure that Ripley considers.[30] Here, making sense of the failure of transitivity of committive consequence essentially involves an appreciation of the *dynamic* character of commitment attribution and acknowledgment. That is, one can *attribute* a set of commitments to some speaker, and if that speaker explicitly *acknowledges* those commitments, this can lead to one's attribution of further commitments that one had not previously attributed. In the case of someone's saying "Bella's a bird," I will take them to be (defeasibly) committed to affirming "Bella flies." If, however, they explicitly acknowledge that commitment, saying "Bella flies," I'm then going to take them to be committed to denying "Bella's a penguin," a commitment I had not previously attributed to them. This is an instance of what Brandom (2018) calls *the consequentiality of explicitation*. Making this inferential phenomenon formally explicit in the way that we have here has required a union of the non-transitive approach associated with Restall-style bilateralism with the scorekeeping model only made available by the framework's featuring Rumfitt-style single conclusion bilateral sequents. It has required, in other words, our having brought these bilateralisms together.

[29] Where λ is the liar sentence, and A and B are arbitrary sentences, Ripley's system allows one to use classical negation and Contraction to derive $A \vdash \lambda$ and $A, \lambda \vdash B$, but, since Ripley's system doesn't contain Cut, one is not able to derive $A \vdash B$ and trivialize the consequence relation.

[30] Reformulated in this context, the principle that Ripley actually rejects is not (what we are calling) Transitivity but, one that is more aptly called, following Restall, *Extensibility*, which lets one move from $\Gamma, \varphi \vdash$ and $\Gamma, \varphi^* \vdash$ to $\Gamma \vdash$. That is, if Γ along with φ is incoherent and Γ along with φ^* is incoherent, then Γ itself must be incoherent. In other words, for any coherent position Γ and any stance φ, Γ must be coherently extensible to contain either φ or it's opposite. Ripley takes the liar sentence to be a counter-example to this principle: a sentence who's affirmation and denial are both out of bounds, relative to any position (even coherent ones). Now, given In and Out, and the logical rules of this system, rejecting Extensibility requires rejecting CT in just the way that rejecting BX does. However, as mentioned above, Ripley would presumably be inclined to reject Out in his approach to the liar, as he presumably does not want to say that one is both committed to affirming the liar and committed to denying the liar.

5 Appendix: Technical Results

Proposition 1: BK is equivalent to Ketonen's sequent calclus K.

Proof: I'll just sketch the proof strategy—it's easy (though a bit tedious) to fill in the details.[31] We show first that that any sequent of the form $\Gamma \vdash \Delta$ in K corresponds to an equivalence class of BK sequents under Reversal of the form $+\langle\Gamma'\rangle, -\langle\Delta\rangle \vdash -\langle\gamma\rangle$, where $\Gamma' = \Gamma \setminus \{\gamma\}$ for any $\gamma \in \Gamma$, and $+\langle\Gamma\rangle, -\langle\Delta'\rangle \vdash +\langle\delta\rangle$ where $\Delta' = \Delta \setminus \{\delta\}$ for any $\delta \in \Delta$. We then induct on proof height to show that this correspondence is preserved across proofs in the two systems (where applications of Reversal in BK are not taken to contribute to proof height). For the base case, we show that the equivalence holds for axioms. For the inductive step, we assume that the correspondence holds up to proof height n, and show that it holds at height $n + 1$ for each of the connective rules. □

Definitions 1.1 and 1.2:

- An inference $\Gamma : \varphi$ is *classically valid* just in case there is no classical valuation v such that all positively signed sentences in Γ are true, all of the negatively signed sentences in Γ are false, and the sentence signed in the formula φ is false if φ is positively signed or true if φ is negatively signed.
- A set Γ is *classically unsatisfiable* just in case there is no classical valuation v such that all positively signed sentences in Γ are true and all of the negatively signed sentences in Γ are false.

Proposition 2: Any classical validity is strictly valid.

Proof: This can be shown by a direct completeness proof, but it's simpler to appeal to the equivalence with K which is known to be complete. It follows from the completeness of K and our translation schema that, for any classically unsatisfiable set Γ, K' derives $\Gamma \vdash$. Accordingly, since, for any classically valid inference $\Gamma : \varphi$, Γ, φ^* is classically unsatisfiable, for any such inference, K' derives $\Gamma, \varphi^* \vdash$, and so, by Proposition 1, BK derives $\Gamma \vdash \varphi$. Since any set of scorekeeping principles π is closed under the rules of BK, any set π contains $\Gamma \vdash \varphi$ and, moreover, contains $\Delta, \Gamma \vdash \varphi$ for any Δ since, if $\Gamma : \varphi$ is classically valid, then $\Delta, \Gamma : \varphi$ is classically valid.

[31] See Simonelli (2022, Appendix and M.S.) for the details.

Accordingly, if $\Gamma : \varphi$ is classically valid, then for every set of moves M, $\varphi \in \pi(\mathrm{M} \cup \Gamma)$, and so $\Gamma \vDash_s \varphi$. □

Definition 2: A rule of the form:

$$\frac{\Gamma_1 : \varphi_1, \Gamma_2 : \varphi_2 \dots \Gamma_n : \varphi_n}{\Delta : \psi}$$

holds with respect to strict/general validity just in case there is no set of scorekeeping principles π such that all of the premises are strictly/generally valid and the conclusion is not strictly/generally valid.

Proposition 3: Monotonicity holds with respect to strict validity, but not general validity

Proof: For strict validity, suppose for contradiction $\Gamma \vDash_s \varphi$ but not $\Gamma, \psi \vDash_s \varphi$. So, for every set of moves M, $\varphi \in \pi(\mathrm{M} \cup \Gamma)$, and there is some set of moves M such that $\varphi \notin \pi(\mathrm{M}' \cup \Gamma \cup \{\psi\})$. But, of course, $\mathrm{M}' \cup \{\psi\}$ is some set of moves, and so $\varphi \in \pi(\mathrm{M}' \cup \Gamma \cup \{\psi\})$. Contradiction, so if $\Gamma \vDash_s \varphi$, then $\Gamma, \psi \vDash_s \varphi$.

For general validity, just suppose $\Gamma \vdash \varphi \in \pi$ but $\Gamma, \psi \vdash \varphi \notin \pi$. Then $\varphi \in \pi(\Gamma)$, and so $\Gamma \vDash_g \varphi$, but $\varphi \notin \pi(\Gamma \cup \{\psi\})$, and so $\Gamma, \psi \nvDash_g \varphi$. □

Proposition 4: Cumulative Transitivity holds with respect to neither general nor strict validity.

Proof: For general validity, just suppose $\Gamma \vdash \varphi \in \pi$ and $\Gamma, \varphi \vdash \psi \in \pi$, but $\Gamma \vdash \psi \notin \pi$. Then $\varphi \in \pi(\Gamma)$, and so $\Gamma \vDash_g \varphi$ and $\psi \in \pi(\Gamma \cup \{\varphi\})$ so $\Gamma, \varphi \vDash_g \psi$, but $\psi \notin \pi(\Gamma)$ so $\Gamma \nvDash_g \psi$. Note that, in this case, $\pi(\Gamma) \neq \pi(\pi(\Gamma))$.

For strict validity, construct the same example, supposing that, for all Δ, $\Delta, \Gamma \vdash \varphi \in \pi$ and $\Delta, \Gamma, \varphi \vdash \psi \in \pi$, but $\Gamma \vdash \psi \notin \pi$. □

References

[1] Ayhan, S. (2021). "Uniqueness of Logical Connectives in a Bilateralist Setting." In I. Sedlár & M. Blicha (Eds.) *The Logica Yearbook 2020* (pp. 1-17). College Publications.

[2] Brandom, R. (1994). *Making It Explicit.* Cambridge, MA: Harvard University Press.

[3] Brandom, Robert. (2000). *Articulating Reasons.* Cambridge, MA: Harvard University Press.

[4] Brandom, R. (2008). *Between Saying and Doing.* Oxford: Oxford University Press.

[5] Brandom, R. (2018). "From Logical Expressivism to Expressivists Logics: Sketch of a Program and Some Implementations." In O. Beran, V. Kolman, & L. Koren (Eds.) *From Rules to Meanings: New Essays on Inferentialism* (pp. 151-164). Routledge.

[6] Del Valle-Inclan, P. and J. Schlöder. (2023). "Coordination and Harmony in Bilateral Logic." *Mind* 132, no. 525: 192-207.

[7] Del Valle-Inclan, P. (2023). "Harmony and Normalization in Bilateral Logic." *Bulletin Section of Logic*.

[8] Dickie, I. (2010.) "Negation, Anti-Realism, and the Denial Defense." *Philosophical Studies* 150: 161-185.

[9] Dummett, M. (1991.) *The Logical Basis of Metaphysics.* Harvard University Press.

[10] Fine, K. (2017) "Truthmaker Semantics", in B. Hale, C. Wright, and A. Miller (eds.) *A Companion to the Philosophy of Language* 2nd edn., 556-577. John Wiley & Sons.

[11] Francez, N. (2014). "Bilateralism in Proof-Theoretic Semantics." *Journal of Philosophical Logic* 43: 239-259.

[12] Francez, N. (2015). *Proof-Theoretic Semantics*. College Publications.

[13] Francez, N. (2019). "Trilateralism, Multilateralism and Poly-Sequents," *Journal of Philosophical Logic* 48, no. 2: 245-262.

[14] Gentzen, G. (1935). "Investigations into Logical Deduction," in *The Collected Papers of Gerhard Gentzen*, ed. M. Szabo, 68-131. Amsterdam: North-Holland. 1969.

[15] Hjortland, O. (2014). "Speech Acts, Categoricity, and the Meanings of Logical Connectives." *Notre Dame Journal of Symbolic Logic*.

[16] Hlobil, U. (2016). "A Nonmonotonic Sequent Calculus for Inferentialist Expressivists." In P. Arazim & M. Dančák (Eds.), *The Logica Yearbook 2015* (pp. 87-105). College Publications.

[17] Hlobil, U. (2018) "Choosing Your Nonmonotonic Logic: A Shopper's Guide." In P. Arazim & T. Lávička (eds.), *The Logica Yearbook 2017* (pp. 109-123). London: College Publications.

[18] Hlobil. U. (2019). "Faithfulness for Naive Valdity." *Synthese* 196, no. 11: 4759-4774.

[19] Hlobil, U. (2023) "The Laws of Thought and the Laws of Truth as Two Sides of One Coin," *Journal of Philosophical Logic* 52:313-343.

[20] Hlobil, U. and R. Brandom. (forthcoming) *Reasons for Logic, Logic for Reasons*, Routledge.

[21] Humberstone, L. "The Revival of Rejective Negation." *Journal of Philosophical Logic*. 29: 331-381.

[22] Incurvati, L., and J. Schlöder. 2017. "Weak Rejection." *Australasian Journal of Philosophy* 95 (4): 741-760.

[23] Incurvati, L., and J. Schlöder. 2019. "Weak Assertion." *Philosophical Quarterly* 69 (277): 741-770.

[24] Incurvati L. and J. Schlöder. (2023). "Inferential Deflationism." *The Philosophical Review*

[25] Kaplan, D. (2018). "A Multi-Succident Sequent Calculus for Logical Expressivists." In P. Arazim & T. Lákiva (Eds.), *The Logica Yearbook 2017* (pp. 139-154). London: College Publications.

[26] Kaplan, D. (2022). *Substructural Content*. Ph.D. Dissertation, University of Pittsburgh.

[27] Ketonen, O. (1944). *Untersuchungen zum Pradikatenkalkul*, Annales Acad. Sci. Fenn. Ser. A.I. 23. Helsinki.

[28] Kürbis, N. (2016). "Some Comments on Ian Rumfitt's Bilateralism." *Journal of Philosophical Logic* 45: 623-644.

[29] Kürbis, N. (2019). *Proof and Falsity: A Logical Investigation.* Cambridge University Press.

[30] Kürbis, N. (2022). "Bilateral Inversion Principle. In *Electronic Proceedings in Theoretical Computer Science 358*, 202-215.

[31] Kürbis, N. (2023). "Supposition: A Problem for Bilateralism." *Bulletin Section of Logic*.

[32] Negri, S. and J. von Plato. (2008). *Structural Proof Theory*. Cambridge: Cambridge University Press.

[33] Nickel, Bernhard. 2013. "Dynamics, Brandom-Style." *Philosophical Studies* 162, no. 2: 333-354.

[34] Restall, G. (2005). "Multiple Conclusions." In P. Hájek, L. Valdés-Villanueva & D. Westerstahl (Eds.), *Logic, Methodology and Philosophy of Science*. College Publications.

[35] Restall, G. (2013). Assertion, Denial, and Non-Classical Theories. In K. Tanaka, F. Berto, E. Mares, & F. Paoli (Eds.), Paraconsistency: Logic and applications (pp. 81–99). Berlin: Springer.

[36] Restall, G. (2016). "Review of *Advances in Proof-Theoretic Semantics*," *Notre Dame Philosophical Reviews.* https://ndpr.nd.edu/news/advances-in-proof-theoretic-semantics

[37] Restall, G. (2020). "Speech Acts & the Question for a Natural Account of Classical Proof."

[38] Ripley, D. (2013). "Paradoxes and Failures of Cut." *Australasian Journal of Philosophy* 91, no. 1: 139-164.

[39] Ripley, D. (2015). "Anything Goes." *Topoi* 34:25-36.

[40] Ripley, David. 2017. "Bilateralism, Coherence, Warrant." In *Act-Based Conceptions of Propositional Content*, ed. F. Moltmann and M. Textor, 307-324. Oxford: Oxford University Press.

[41] Rosenblatt, L. 2019. "Bilateralism and Invalidities." *Inquiry*.

[42] Rumfitt, Ian. (2000). "Yes and No." *Mind* 109, no. 436: 781-823.

[43] Rumfitt, Ian. 2008. "Knowledge by Deduction." *Grazer Philosophische Studien* 77: 61-84.

[44] Rumfitt, I. 2015. *The Boundary Stones of Thought*. Oxford: Oxford University Press.

[45] Sellars, W. 1974. "Meaning as Functional Classification." *Synthese*

[46] Simonelli, Ryan. 2022. *Meaning and the World*. Ph.D. Dissertation. University of Chicago.

[47] Simonelli, Ryan. M.S. "A General Schema for Bilateral Proof Rules." Available at ryansimonelli.com/papers.

[48] Smiley, T. 1996. "Rejection." *Analysis* 56, no. 1: 1-9.

[49] Shoesmith, D.J. and T.J. Smiley. 1978. *Multiple-Conclusion Logic*. Cambridge: Cambridge University Press.

[50] Stalnaker, Robert. 1978. "Assertion." *Syntax and Semantics* 9: 315-332. Reprinted in Stalnaker, Robert. 1999. *Context and Content*, 78-95.

[51] Steinberger, Florian. 2011. "Why Conclusions Should Remain Single." *Journal of Philosophical Logic* 40: 333-355.

[52] Steinberger, F. 2011b. "What Harmony Could and Could Not Be." *Australasian Journal of Philosophy* 89:4, 617-639.

[53] Tarski, A. 1930. "Über einige fundamentale Begriffe der Metamathematik." Reprinted in English as "On Some Fundamental Concepts of Metamathematics" in Taski, A. 1956. *Logic, Semantics, Metamathematics*. Oxford: Clarendon Press.

[54] Tanter, Kai. 2021. "Subatomic Inferences: an Inferentialist Semantics for Atomics, Predicates, and Names." *Review of Symbolic Logic.*

[55] Veltman, Frank. 1996. "Defaults in Update Semantics." *Journal of Philosophical Logic* 25, 221-261.

[56] Wansing, H. (2016.) "Falsification, Natural Deduction and Bi-Intuitionistic Logic." *Journal of Logic and Computation* 26, no. 1: 425-450.