Generalized Bilateral Harmony

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Abstract: I introduce a new schematic notation for formulating bilateral natural deduction systems, and I use this notation to formulate three distinct bilateral natural deduction systems for classical logic. I then propose a new criterion for bilateral harmony that I argue is superior to the existing criteria proposed in the literature. Finally, I show, at the schematic level, that all three bilateral systems meet this criterion of bilateral harmony.

Keywords: bilateralism, natural deduction, harmony

1 Introduction

Classical natural deduction famously suffers from a lack of harmony between the introduction and elimination rules. In response to this issue, Rumfitt (2000) argues that if one wants to account for the meanings of the classical connectives in terms of the rules governing their use in a natural deduction system, one should opt for a *bilateral* system, in which formulas are positively or negatively signed, expressing affirmations or denials. Such systems straightforwardly resolve the lack of harmony between introduction and elimination rules. However, they give rise to the further concern of lack of harmony between the positive and negative rules in the system, and some authors, such as Gabbay Gabbay (2017), have argued that this is a serious problem for bilateralism. In recent literature on bilateralism, several different proposals for bilateral harmony have been put forth in response to this concern (Francez, 2014a, Kürbis, 2022, del Valle-Inclan & Schlöder, 2023, del Valle-Inclan, 2023, Kürbis, 2021). I contend here, however, that all such proposals are unsatisfactory, either failing to rule out disharmonious connectives, ruling out harmonious ones, or achieving extensional adequacy at the expense of being *ad hoc*. This paper introduces a novel condition for bilateral harmony that is sufficient, necessary, and conceptually well-motivated, framing it within a broader, generalized approach to bilateralism. This generalized approach treats bilateral systems with a notation

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that schematizes over the polarity of positive or negative signs, enabling us to abstract away from specific bilateral rules and consider instead general bilateral rule forms and their various proof-theoretic virtues or vices.

This paper is structured as follows. In Section Two, I introduce a schematic notation that enables a specification of a bilateral system for all of the classical connectives in terms of a single rule schema. This enables me to schematically specify three distinct bilateral natural deduction systems for classical logic. In Section Three, I introduce the notions of unilateral and bilateral harmony as a pair of constraints that the rules of any bilateral natural deduction system must meet. In Section Four, I criticize the three existing approaches to bilateral harmony in the literature. Finally, in Section Five, I propose a new criterion of bilateral harmony and prove, at the schematic level, that all three systems I've specified meet this new criterion of bilateral harmony whereas problematic connectives, such as the bilateral version of *tonk*, fail to meet it.

2 Three Schematic Systems

There are two key innovations of bilateral natural deduction systems for classical logic of the sort proposed by Smiley (1996) and Rumfitt (2000).[2] The first key innovation is the rules for negation. In Rumfitt's system, they are the following:

$$\frac{-\langle\varphi\rangle}{+\langle\neg\varphi\rangle}\ +_{\neg I} \qquad\qquad \frac{+\langle\neg\varphi\rangle}{-\langle\varphi\rangle}\ +_{\neg E}$$

$$\frac{+\langle\varphi\rangle}{-\langle\neg\varphi\rangle}\ -_{\neg I} \qquad\qquad \frac{-\langle\neg\varphi\rangle}{+\langle\varphi\rangle}\ -_{\neg E}$$

These rules jointly codify that denying a sentence has the same logical significance as asserting its negation. They are obviously harmonious, and they clearly define an involutive negation, as both double negation introduction and

[2] Here, I limit my attention to bilateral natural deduction systems of the sort proposed by Smiley (1996) and Rumfitt (2000), primarily in the context of classical logic, in which formulas are positively and negatively signed. The authors discussed here all develop this style of bilateralism. In the past several years, a different style of bilateral system has been developed, primarily in the context of intuitionistic logic, by Wansing (2013), Wansing (2017), Ayhan (2021), Drobyshevich (2019), Wansing and Ayhan (2023), and others which involves a signing of the *turnstile* (or the horizontal deduction line) to express verification or falsification. There are interesting and important questions to ask about the relationship between these two styles of bilateralism. However, addressing those questions is left for another paper.

elimination are immediately derived through two applications of the I-rules or E-rules respectively.

The second key innovation of Smiley/Rumfitt-style bilateral natural deduction systems are the *coordination principles*, structural rules which "coordinate" the opposite stances of affirmation and denial, formally codifying the sense in which these stances really are opposites. The standard formulation of bilateralism for classical logic, owed to Rumfitt, contains the following two, which I call "Incoherence" and "Reductio":[3]

$$\frac{A \quad A^*}{\bot}\ \text{Incoherence} \qquad\qquad \begin{array}{c} \overline{A}^{\,u} \\ \vdots \\ \dfrac{\bot}{A^*}\ \text{Reductio}^{\,u} \end{array}$$

Here, A is any signed sentence, and starring a signed sentence yields the oppositely signed sentence. Thus, Incoherence says that from some stance A and its opposite A^*, one can conclude incoherence, and Reductio says that if, given the assumption of some stance A, one can conclude incoherence, then one can discharge that assumption and conclude the opposite stance, A^*. With these coordination principles, the negation rules given above yield classical negation.

This paper brings a third key idea to bilateralism.[4] Bilateral notation enables us to think of the rules for all of the binary connectives of classical logic as instances of general rule schemas. This enables us to consider different sets of binary connective rules and their respective proof-theoretic virtues and vices at a higher level of generality than standard approaches, abstracting from the polarity of signs (whether they are positive or negative) and just considering the opposition between stances towards sentences. To do this, I deploy a notation

[3]Smiley's original formulation of bilateralism for classical logic involved just one principle, which Rumfitt calls "Smiliean Reductio." There are other ways of specifying the coordination principles for classical logic. For instance, del Valle-Inclan (2023) proposes "Bilateral Explosion" and "Bilateral Excluded Middle." However, most recent proponents of bilateralism for classical logic (e.g. Kürbis (2021), Hjortland (2014), del Valle-Inclan and Schlöder (2023), Incurvati and Schlöder (2023)) have followed Rumfitt in using these two principles, and I will do so here. Given the inter-derivability of different equivalent sets of coordination principles, the main proposal for bilateral harmony presented in this paper can be implemented in systems containing different coordination principles.

[4]A similar idea has been developed in the context of signed tableaux systems by Smullyan (1968), which are themselves a kind of bilateral system. However, the version of the schematic approach adopted here is both more flexible and more conceptually transparent.

that schematizes over signs, using variables such as $\boldsymbol{a}$ and $\boldsymbol{b}$ to indicate signs that may be either $+$ or $-$ along with a function * that maps $+$ to $-$ and $-$ to $+$. So, for any signed formula of the form $\boldsymbol{a}\langle\varphi\rangle$, where $\boldsymbol{a} \in \{+, -\}$, if $\boldsymbol{a} = +$ then $\boldsymbol{a}^* = -$, and if $\boldsymbol{a} = -$ then $\boldsymbol{a}^* = +$ (and so $\boldsymbol{a}^{**} = \boldsymbol{a}$).[5] With this notation introduced, we can consider rule sets as a whole in terms of their general form and do our proof-theory at this higher level of generality.

There are three bilateral systems I will consider here. All are extensions of the bilateral system originally proposed by Smiley (1996). Smiley proposes a bilateral system with just rules for negation, conjunction and disjunction. The rules for conjunction and disjunction both take the following form:

$$\frac{\boldsymbol{a}\langle\varphi\rangle \quad \boldsymbol{b}\langle\psi\rangle}{\boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_{\circ\mathrm{I}} \qquad\qquad \frac{\boldsymbol{c}\langle\varphi \circ \psi\rangle}{\boldsymbol{a}\langle\varphi\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_1} \qquad\qquad \frac{\boldsymbol{c}\langle\varphi \circ \psi\rangle}{\boldsymbol{b}\langle\psi\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_2}$$

Understanding the horizontal line as expressing commitment (e.g. Brandom, 1994, Incurvati & Schlöder, 2023), the $\boldsymbol{c}_{\circ\mathrm{I}}$ rule says that if one takes stance $\boldsymbol{a}$ to φ and stance $\boldsymbol{b}$ to ψ, then one is committed to taking stance $\boldsymbol{c}$ to $\varphi \circ \psi$. The $\boldsymbol{c}_{\circ\mathrm{E}}$ rules says that if one takes stance $\boldsymbol{c}$ to $\varphi \circ \psi$, then one is committed to taking stance $\boldsymbol{a}$ to φ and one is also committed to taking stance $\boldsymbol{b}$ to ψ. Though Smiley provides rules only for conjunction and disjunction that take this form, the whole set of standard binary connectives, along with several not so standard ones (the Sheffer Stroke, Perice's arrow, and the dual of the conditional), can be given rules of this form:

$$\begin{array}{ll}
\wedge\text{: } \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = + & \vee\text{: } \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = - \\
|\text{: } \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = - & \downarrow\text{: } \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = + \\
\rightarrow\text{: } \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = - & \succ\!\!-\text{: } \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = + \\
-\!\!<\text{: } \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = + & \leftarrow\text{:} \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = -
\end{array}$$

The distinction between the rules for conjunction and disjunction and all of the other rules is that the conjunction and disjunction rules are *bilaterally homogeneous*, each containing only one sign, positive or negative, whereas all of the other rules are *bilaterally mixed*, containing both positive and negative signs. In a unilateral context, such "mixed" form would require appeal to negation, and so would violate the criterion of separability among the connective rules. In a bilateral context, however, there is no reason not to allow bilaterally mixed rules, and so this enables us to put forward rules for all of the connectives in

[5] So that the star is not ambiguous, we might now say that, where A is shorthand for a formula of the form $\boldsymbol{a}\langle\varphi\rangle$, A^* is shorthand for $\boldsymbol{a}^*\langle\varphi\rangle$.

terms of a single rule schema. I'll call this system, where rules for all of the connectives are given by this schema, BNK0.

It is easy to show, as Smiley does for the conjunctive and disjunctive fragment of this system, that, given the coordination principles, BNK0 is a sound and complete system for classical propositional logic containing all of these connectives. Still, although, in a *truth-functional* sense BNK0 is complete, in a *proof-theoretic* sense, it is not complete, since, for each connective, it contains only rules for affirming or denying that connective. A proof-theoretically complete bilateral system must include rules for *both* affirming and denying each connective. I now want to consider three systems that complete BNK0 with rules specifying the grounds for and consequences of taking the opposite stance, $\boldsymbol{c}^*$, to $\varphi \circ \psi$.

The first system I'll consider here is based on Rumfitt's (2000) system. Rumfitt supplements Smiley's positive conjunction and negative disjunction rules with negative conjunction and positive disjunction rules of the following form:

$$\frac{\boldsymbol{a}^*\langle\varphi\rangle}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\ \boldsymbol{c}^*{}_{\circ \mathrm{I}_1} \qquad \frac{\boldsymbol{b}^*\langle\psi\rangle}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\ \boldsymbol{c}^*{}_{\circ \mathrm{I}_2} \qquad \frac{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle \quad \begin{matrix}\overline{\boldsymbol{a}^*\langle\varphi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{matrix} \quad \begin{matrix}\overline{\boldsymbol{b}^*\langle\psi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{matrix}}{A}\ \boldsymbol{c}^*{}_{\circ \mathrm{E}}$$

Now, in the system proposed by Rumfitt, the rules for the conditional are different in form from the rules for conjunction and disjunction. In the context of bilateral classical logic, however, there is no reason for this difference. One way to see this is to see that the negative conditional rules proposed by Rumfitt are the following:

$$\frac{+\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi\to\psi\rangle}\ -_{\to \mathrm{I}} \qquad \frac{-\langle\varphi\to\psi\rangle}{+\langle\varphi\rangle}\ -_{\to \mathrm{E}_1} \qquad \frac{-\langle\varphi\to\psi\rangle}{-\langle\psi\rangle}\ -_{\to \mathrm{E}_2}$$

These are of exactly the same form as the positive conjunction and negative disjunction rules he provides; they are of the form of the $\boldsymbol{c}_\circ$ rules above. Insofar as this a uniform specification of the conditions and consequences of taking *one* stance (be it positive or negative) towards a conjunction, disjunction, or conditional, it's reasonable to think that the specification of the conditions and consequences to taking the *opposite* stance (be it negative or positive) towards a conjunction, disjunction, or conditional, should likewise be uniform. BNK1 provides such a uniform specification. It should clear, however, that it's not the only one.

Rather than providing rules that take the form of conjunction and disjunction in Rumfitt's system, we could alternatively provide a system in which the rules for all of the connectives have the form of Rumfitt's rules for the conditional. del Valle-Inclan and Schlöder (2023) have recently proposed such a system. In del Valle-Inclan and Schlöder's system, the $\boldsymbol{c}_\circ$ rules are, once again, those of BNK0. So, the system contains the same same positive conjunction, negative disjunction, and negative conditional rules. However, the $\boldsymbol{c}^*$ rules are the following:

$$\dfrac{\begin{array}{c}\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,u}\\ \vdots\\ \overline{\boldsymbol{b}^*\langle\psi\rangle}\end{array}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\;\boldsymbol{c}^*{\circ_{\mathrm{I}}}^{u} \qquad\qquad \dfrac{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle\quad \boldsymbol{a}\langle\varphi\rangle}{\boldsymbol{b}^*\langle\psi\rangle}\;\boldsymbol{c}^*{\circ_{\mathrm{E}}}$$

Following Rumfitt's rules for the conditional, del Valle-Inclan and Schlöder treat introduction rule with the hypothetical proof from $\boldsymbol{b}\langle\psi\rangle$ to $\boldsymbol{a}^*\langle\varphi\rangle$ and the elimination rule concluding $\boldsymbol{a}^*\langle\varphi\rangle$ from $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$ and $\boldsymbol{b}\langle\psi\rangle$ as derived.[6] One can alternately treat these additional rules as basic. Either way, I'll call this second system, BNK2.

Finally, I want to introduce a third bilateral system, with rules of a form that have not been previously considered, at least in a bilateral context.[7] Once again $\boldsymbol{c}_\circ$ rules are kept as is, but we now use the following $\boldsymbol{c}^*{}_\circ$ rules:

[6]The derivations go as follows:

$$\dfrac{\dfrac{\dfrac{\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,2}\qquad \begin{array}{c}\overline{\boldsymbol{b}\langle\psi\rangle}^{\,1}\\ \mathcal{D}\\ \overline{\boldsymbol{a}^*\langle\varphi\rangle}\end{array}}{\bot}\;\text{Inc.}}{\boldsymbol{b}^*\langle\psi\rangle}\;\text{Red.}^1}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\;\boldsymbol{c}^*{\circ_{\mathrm{I}}}^{2} \qquad\qquad \dfrac{\dfrac{\dfrac{\boldsymbol{c}\langle\varphi\circ\psi\rangle\quad \overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1}}{\boldsymbol{b}^*\langle\psi\rangle}\;\boldsymbol{c}^*{\circ_{\mathrm{E}}}\qquad \boldsymbol{b}^*\langle\varphi\rangle}{\bot}\;\text{Inc.}}{\boldsymbol{a}^*\langle\varphi\rangle}\;\text{Red.}^1$$

[7]Murzi (2020) considers such rules in a unilateral context, where they suffer from a problem of separability. There is no such problem in a bilateral context.

$$\dfrac{\dfrac{\begin{matrix}\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,u} & \overline{\boldsymbol{b}\langle\psi\rangle}^{\,v} \\ \vdots & \vdots \end{matrix}}{\bot}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\ \boldsymbol{c}^*{}_{\circ\mathrm{I}}{}^{u,v} \qquad \dfrac{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle \quad \boldsymbol{a}\langle\varphi\rangle}{\boldsymbol{b}^*\langle\psi\rangle}\ \boldsymbol{c}^*{}_{\circ\mathrm{E}_1} \qquad \dfrac{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle \quad \boldsymbol{b}\langle\psi\rangle}{\boldsymbol{a}^*\langle\varphi\rangle}\ \boldsymbol{c}^*{}_{\circ\mathrm{E}_2}$$

Thus, we have the same elimination rules as BNK2 (I'll treat the second as primitive here), but a distinct introduction rule. I'll call this third system BNK3. The positive and negative introduction rules of BNK3, specified at this schematic level, fit together very intuitively. Once again, the $\boldsymbol{c}_{\circ I}$ rule says that one is committed to taking stance $\boldsymbol{c}$ to $\varphi \circ \psi$ if one takes stance $\boldsymbol{a}$ to φ and stance $\boldsymbol{b}$ to ψ. The $\boldsymbol{c}^*{}_{\circ I}$ rule, on the other hand, says that one is committed to taking the opposite stance, $\boldsymbol{c}^*$, to $\varphi \circ \psi$ if taking $\boldsymbol{a}$ to φ along with taking stance $\boldsymbol{b}$ to ψ is incoherent.

So, to sum up, BNK0 provides rules for taking one stance, $\boldsymbol{c}$, to each of the connectives. Whether this stance is positive or negative for any given connective is determined by the assignment of signs for connectives given above. All of the rules for taking stance $\boldsymbol{c}$ towards $\varphi \circ \psi$ have the same basic form as the familiar positive conjunction rules. The three systems considered here each supplement these $\boldsymbol{c}_\circ$ rules with a set of rules for taking the opposite stance, $\boldsymbol{c}^*$, towards $\varphi \circ \psi$. In particular, I've specified the following three systems:

1. **BNK1:** BNK0's $\boldsymbol{c}_\circ$ rules + BNK1's $\boldsymbol{c}^*{}_\circ$ rules (which all have the form of the familiar positive disjunction rules)

2. **BNK2:** BNK0's $\boldsymbol{c}_\circ$ rules + BNK2's $\boldsymbol{c}^*{}_\circ$ rules (which all have the form of the familiar positive conditional rules)

3. **BNK3:** BNK0's $\boldsymbol{c}_\circ$ rules + BNK3's $\boldsymbol{c}^*{}_\circ$ rules (which all have the new form shown above)

3 Unilateral and Bilateral Harmony

Having laid out these three systems, I now turn to the issue of the *harmony* among their rules. Let us start by first considering unilateral harmony, between introduction and elimination rules. Unilateral harmony means that the introduction and elimination rules are neither too strong nor too weak relative to each other. The classic case of disharmonious rules are those for the connective *tonk*, proposed first by Prior (1967), which has the following (positive) introduction and elimination rules:

$$\frac{+\langle\varphi\rangle}{+\langle\varphi \textit{ tonk } \psi\rangle} +_{\textit{tonk}\,\mathrm{I}_1} \qquad \frac{+\langle\psi\rangle}{+\langle\varphi \textit{ tonk } \psi\rangle} +_{\textit{tonk}\,\mathrm{I}_2}$$

$$\frac{+\langle\varphi \textit{ tonk } \psi\rangle}{+\langle\varphi\rangle} +_{\textit{tonk}\,\mathrm{E}_1} \qquad \frac{+\langle\varphi \textit{ tonk } \psi\rangle}{+\langle\psi\rangle} +_{\textit{tonk}\,\mathrm{E}_2}$$

The problem with *tonk* is that the elimination rules are too strong relative to the introduction rules. As such, it trivializes the logic, enabling us to conclude $+\langle\psi\rangle$ from $+\langle\varphi\rangle$ for arbitrary φ and ψ. A set of rules with the opposite problem are those for the connective that Francez (2015) calls *tunk*:

$$\frac{+\langle\varphi\rangle \quad +\langle\psi\rangle}{+\langle\varphi \textit{ tunk } \psi\rangle} +_{\textit{tunk}\,\mathrm{I}} \qquad \frac{+\langle\varphi \textit{ tunk } \psi\rangle \quad \begin{array}{c}\overline{+\langle\varphi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{array} \quad \begin{array}{c}\overline{+\langle\psi\rangle}^{\,v}\\ \vdots \\ \overline{A}\end{array}}{A} +_{\textit{tunk}_E}{}^{u,v}$$

Here, the elimination rule is *too weak* relative to the introduction rule. Though introducing a connective with these rules does not trivialize the consequence relation like introducing *tonk*, these rules are nevertheless disharmonious in an obvious sense, and a criterion for harmony ought to rule them out. So, any criterion of unilateral harmony ought to rule out *tonk* and *tunk* and do so in a systematic way.

A now standard approach to unilateral harmony, formulated by Pfenning and Davies (2001) expanding on a key idea of Prawitz (1965), is to conceive of harmony as established by a *reduction*, showing that the only consequences you can derive from a complex formula are among the grounds you used to derive it, and an *expansion* showing that, by extracting consequences from a complex formula, you can always recover the grounds required to derive it. The reduction shows that the elimination rules are not *too strong* relative to the introduction rules, whereas the expansion shows that the elimination rules are not *too weak* relative to the introduction rules. Schematically, these reductions and expansions for our $\boldsymbol{c}_\circ$ rules, establishing unilateral harmony, go as follows:

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{a}\langle\varphi\rangle\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \boldsymbol{b}\langle\psi\rangle\end{array}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\boldsymbol{c}_{\circ\mathrm{I}}}{\boldsymbol{a}\langle\varphi\rangle}\boldsymbol{c}_{\circ\mathrm{E}_1} \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{a}\langle\varphi\rangle\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \boldsymbol{b}\langle\psi\rangle\end{array}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\boldsymbol{c}_{\circ\mathrm{I}}}{\boldsymbol{b}\langle\psi\rangle}\boldsymbol{c}_{\circ\mathrm{E}_2} \quad \rightsquigarrow_r \quad \begin{array}{c}\mathcal{D}_1\\ \boldsymbol{a}\langle\varphi\rangle\end{array} \quad \begin{array}{c}\mathcal{D}_2\\ \boldsymbol{b}\langle\psi\rangle\end{array}$$

$$\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array} \quad \rightsquigarrow_e \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array}}{\boldsymbol{a}\langle\varphi\rangle}\,\boldsymbol{c}_{\circ\mathrm{E}_1} \qquad \dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array}}{\boldsymbol{b}\langle\psi\rangle}\,\boldsymbol{c}_{\circ\mathrm{E}_2}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}_{\circ\mathrm{I}}$$

This criterion of unilateral harmony enables us to rule out *tonk* and *tunk* and do so in a systematic way that reveals the sense in which they have the opposite problem: for *tonk*, no reduction is possible, whereas, for *tunk*, no expansion is possible.

The $\boldsymbol{c}^*$ rules of BNK1, BNK2, and BNK3 are all unilaterally harmonious as well. The forms of the $\boldsymbol{c}^*$ rules of BNK1 and BNK2 are familiar as those of the standard unilateral disjunction rules and conditional rules respectively, and the well-known reductions and expansions for those rules can be schematized to yield unilateral harmony proofs for BNK1 and BNK2. For BNK3, the reductions and expansions go as follows:

$$\dfrac{\dfrac{\begin{array}{c}\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1} \quad \overline{\boldsymbol{b}\langle\psi\rangle}^{\,2}\\ \mathcal{D}_1\\ \bot\end{array}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{}_{\circ\mathrm{I}}^{1,2} \qquad \begin{array}{c}\mathcal{D}_3\\ \boldsymbol{a}\langle\varphi\rangle\end{array}}{\boldsymbol{b}^*\langle\psi\rangle}\,\boldsymbol{c}^*{}_{\circ\mathrm{E}_1} \quad \rightsquigarrow_r \quad \dfrac{\begin{array}{c}\mathcal{D}_3 \qquad\quad\\ \boldsymbol{a}\langle\varphi\rangle \quad \overline{\boldsymbol{b}\langle\psi\rangle}^{\,1}\\ \mathcal{D}_1\\ \bot\end{array}}{\boldsymbol{b}^*\langle\psi\rangle}\,\text{Reductio}^1$$

$$\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}^*\langle\varphi\circ\psi\rangle\end{array} \quad \rightsquigarrow_e \quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}^*\langle\varphi\circ\psi\rangle\end{array} \quad \overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1}}{\boldsymbol{b}^*\langle\psi\rangle}\,\boldsymbol{c}^*{}_{\circ\mathrm{E}_1} \qquad \overline{\boldsymbol{b}\langle\psi\rangle}^{\,2}}{\bot}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{}_{\circ\mathrm{I}}^{1,2}$$

So, all three systems are unilaterally harmonious. In a bilateral context, however, unilateral harmony between introduction and elimination rules is not enough.

To see why a further criterion of bilateral harmony is needed to supplement a criterion of unilateral harmony, consider the rules for the connective that I'll call *bonk*, presented first in the form of BNK1 rules:

$$\dfrac{+\langle\varphi\rangle}{+\langle\varphi\ \textit{bonk}\ \psi\rangle}\,+_{\textit{bonk}\,\mathrm{I}} \qquad\qquad \dfrac{+\langle\psi\rangle}{+\langle\varphi\ \textit{bonk}\ \psi\rangle}\,+_{\textit{bonk}\,\mathrm{I}}$$

$$\dfrac{+\langle \varphi\ \mathit{bonk}\ \psi\rangle \qquad \begin{matrix}\overline{+\langle\varphi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{matrix} \qquad \begin{matrix}\overline{+\langle\psi\rangle}^{\,v}\\ \vdots\\ \overline{A}\end{matrix}}{A}\ {+_{\mathit{bonk}_E}}^{u,v}$$

$$\dfrac{-\langle\varphi\rangle}{-\langle \varphi\ \mathit{bonk}\ \psi\rangle}\ -_{\mathit{bonk}\ \mathrm{I}} \qquad\qquad \dfrac{-\langle\psi\rangle}{-\langle \varphi\ \mathit{bonk}\ \psi\rangle}\ -_{\mathit{bonk}\ \mathrm{I}}$$

$$\dfrac{-\langle \varphi\ \mathit{bonk}\ \psi\rangle \qquad \begin{matrix}\overline{-\langle\varphi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{matrix} \qquad \begin{matrix}\overline{-\langle\psi\rangle}^{\,v}\\ \vdots\\ \overline{A}\end{matrix}}{A}\ {-_{\mathit{bonk}_E}}^{u,v}$$

These rules, which combine BNK1's positive disjunction rules with its negative conjunction rules, are unilaterally harmonious. Insofar as a harmony constraint is supposed to rule out *tonkish* connectives, there must be failure of bilateral harmony, for rules of this form, like *tonk*, let us conclude both $+\langle q\rangle$ from $+\langle p\rangle$ for arbitrary atomics p and q.[8]

$$\dfrac{\dfrac{+\langle p\rangle}{+\langle p\ \mathit{bonk}\ q\rangle} \qquad \dfrac{\overline{-\langle q\rangle}^{\,1}}{-\langle p\ \mathit{bonk}\ q\rangle}}{\dfrac{\perp}{+\langle q\rangle}}$$

Note here that only the introductions rules are used to derive $+\langle q\rangle$ from $+\langle p\rangle$. The problem with *bonk*, and other connectives with rules of this same form, is that the grounds for concluding opposite stances towards φ *bonk* ψ specified by the introduction rules are *too weak* relative to each other. A criterion of bilateral harmony, then, must rule out these rules as disharmonious.

The opposite of *bonk* is a connective that Kürbis (2021) calls *conk*, which combines the positive conjunction rules with the negative disjunction rules:

[8] In similar fashion, they also let us conclude $-\langle p\rangle$ from $+\langle q\rangle$, $-\langle q\rangle$ from $-\langle p\rangle$, and $+\langle p\rangle$ from $+\langle q\rangle$.

$$\frac{+\langle\varphi\rangle \quad +\langle\psi\rangle}{+\langle\varphi \mathbin{\textit{conk}} \psi\rangle}\ +_{\textit{conk}\ \mathrm{I}} \qquad \frac{+\langle\varphi \mathbin{\textit{conk}} \psi\rangle}{+\langle\varphi\rangle}\ +_{\textit{conk}\ \mathrm{E}_1} \qquad \frac{+\langle\varphi \mathbin{\textit{conk}} \psi\rangle}{+\langle\psi\rangle}\ +_{\textit{conk}\ \mathrm{E}_2}$$

$$\frac{-\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi \mathbin{\textit{conk}} \psi\rangle}\ -_{\textit{conk}\ \mathrm{I}} \qquad \frac{-\langle\varphi \mathbin{\textit{conk}} \psi\rangle}{-\langle\varphi\rangle}\ -_{\textit{conk}\ \mathrm{E}_1} \qquad \frac{-\langle\varphi \mathbin{\textit{conk}} \psi\rangle}{-\langle\psi\rangle}\ -_{\textit{conk}\ \mathrm{E}_2}$$

Here, once again, we have unilateral harmony, but not bilateral harmony. In this case, however, grounds for introducing opposite stances to φ *conk* ψ, specified by the introduction rules, are *too strong* relative to each other, and so, rather than using the introduction rules, we derive $+\langle q\rangle$ from $+\langle p\rangle$ using only the elimination rules:

$$\cfrac{\cfrac{\cfrac{\cfrac{}{-\langle p \mathbin{\textit{conk}} q\rangle}\ ^{1}}{-\langle p\rangle}\ -_{\textit{conk}\ \mathrm{E}_1} \qquad +\langle p\rangle}{\cfrac{\bot}{+\langle p \mathbin{\textit{conk}} q\rangle}}\ \mathrm{Inc.}}{+\langle q\rangle}\ +_{\textit{conk}\ \mathrm{E}_2}$$

So, a criterion of bilateral harmony must rule out these rules as disharmonious as well. In the next section, I'll consider the three approaches that exist in the literature for ruling out connectives like *bonk* and *conk*.[9]

4 Three Approaches to Bilateral Harmony

The first approach to bilateral harmony is owed to Francez (2014, 2015), and a variant of this approach has also been put forward by Kürbis (2022). Francez's criterion of bilateral harmony essentially works on the assumption, shared by Rumfitt (2000), that the starting point for a bilateral system ought to be a unilateral system of the sort proposed by Gentzen (1935). Given positive introduction rules of the form proposed by Gentzen (combining rules in the case of conjunction, splitting rules in the case of disjunction, and hypothetical rules, in the case of the conditional), Francez provides a recipe for determing corresponding negative rules, and his criterion of bilateral harmony is simply that the rules conform to this recipe. Francez's criterion rules out the rules for

[9]There are a number of other bilaterally tonkish connectives that have been proposed in the literature (del Valle-Inclan and Schölder's *blink* and *bink*, Gabbay's (2017) $\bullet$), but *bonk* and *conk* are representative examples of the two basic ways in which a connective can be bilaterally disharmonious.

conk and *bonk*. For *conk*, since the positive introduction rule is a combining rule with two premises, the corresponding negative introduction rules must be two splitting rules, each one having, as its lone premise, the opposite of one of the premises of the positive introduction rule. For *bonk*, since the positive introduction rules are two splitting rules, the negative introduction rule must be a combining rule with two premises, each the opposite of the corresponding splitting rule.

Now, one basic conceptual problem with Francez's approach, noted by Kürbis (2021), is that the privileging of rules for affirmation, deriving rules for denial by "inversion," is antithetical to the basic philosophical commitment of bilateralism of treating affirmation and denial on a par with each other. However, even bracketing this conceptual problem, a more concrete problem is that it simply fails to provide an adequate general criterion of bilateral harmony. On the one hand, it simply rules out, without any justification, any of the rules of BNK2 where $\boldsymbol{c} = +$ and all of the rules of BNK3. For a concrete case, consider BNK3's positive and negative introduction rules for conjunction:

$$\dfrac{+\langle\varphi\rangle \quad +\langle\psi\rangle}{+\langle\varphi \wedge \psi\rangle}\,+_{\wedge\mathrm{I}} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\overline{+\langle\varphi\rangle}^{\,u} & \overline{+\langle\psi\rangle}^{\,v} \\ \vdots & \vdots\end{matrix}}{\bot}}{-\langle\varphi \wedge \psi\rangle}\,-_{\wedge\mathrm{I}}$$

These rules clearly *seem* harmonious. If one is going to claim that they are not harmonious, one ought to have a good argument for this claim. Francez's criterion of harmony simply rules them out by fiat. Even worse, consider positive and negative introduction rules for the connective *bonk*, in the form of the BNK3 rules:

$$\dfrac{\dfrac{\begin{matrix}\overline{-\langle\varphi\rangle}^{\,u} & \overline{-\langle\psi\rangle}^{\,v} \\ \vdots & \vdots\end{matrix}}{\bot}}{+\langle\varphi \textit{ bonk } \psi\rangle}\,{+_{bonk\mathrm{I}}}^{u,v} \qquad\qquad \dfrac{\dfrac{\begin{matrix}\overline{+\langle\varphi\rangle}^{\,u} & \overline{+\langle\psi\rangle}^{\,v} \\ \vdots & \vdots\end{matrix}}{\bot}}{-\langle\varphi \textit{ bonk } \psi\rangle}\,{-_{bonk\mathrm{I}}}^{u,v}$$

We can use these introduction rules and the coordination principles to derive $+\langle q\rangle$ from $+\langle p\rangle$ for arbitrary p and q:[10]

[10]Note here, we use vacuous discharges, as is standard in classical natural deduction systems.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{-\langle p\rangle}^{\,1} \qquad +\langle p\rangle}{\bot}\,\text{Inc.}}{+\langle p\ \mathit{bonk}\ q\rangle}\,{+_{\mathit{bonk}\mathrm{I}}}^{1,0} \qquad \dfrac{\dfrac{\overline{+\langle q\rangle}^{\,2} \qquad \overline{-\langle q\rangle}^{\,3}}{\bot}}{-\langle p\ \mathit{bonk}\ q\rangle}\,{-_{\mathit{bonk}\mathrm{I}}}^{2,0}}{\bot}\,\text{Inc.}}{+\langle q\rangle}$$

Clearly, then, these rules are just as disharmonious as the above rules for *bonk*. Francez's criterion of bilateral harmony, however, is simply silent on whether these rules are harmonious or not, since he simply doesn't consider bilateral rules of this form. One might add some further criteria to cover rules of this form, but this seems hopelessly ad hoc, and won't ward against potential bilateral rules of yet different forms. A more general and principled approach is needed.

I now turn to a second approach to bilateral harmony, recently been proposed by del Valle-Inclan and Schlöder (2023). The approach begins by noting that, in the trivializing proofs above using *bonk* and *conk*, we use the coordination principles of Reductio and Incoherence on logically complex formulas. del Valle-Inclan and Schlöder's proposed bilateral harmony constraint, aimed at ruling out such problematic connectives, is to require that all coordination principles can be restricted to atomics. Notably, as Ferreira (2008) points out, BNK1 *fails* to meet this constraint; Reductio in particular cannot be restricted to atomics. To see this, consider the proof of the law of non-contradiction:

$$\dfrac{\dfrac{\dfrac{\overline{+\langle p\wedge\neg p\rangle}^{\,1}}{+\langle p\rangle}\,+_{\wedge\mathrm{E}_2} \qquad \dfrac{\dfrac{\overline{+\langle p\wedge\neg p\rangle}^{\,1}}{+\langle\neg p\rangle}\,+_{\wedge\mathrm{E}_2}}{-\langle p\rangle}\,+_{\neg\mathrm{E}}}{\bot}\,\text{Inc.}}{-\langle p\wedge\neg p\rangle}\,\text{Red.}^{1}$$

Here, we use Reductio on $+\langle p\wedge\neg p\rangle$, deriving an incoherence on the basis of this assumption and concluding the opposite, $-\langle p\wedge\neg p\rangle$. As Ferreira shows, there is no way to derive this formula without such a use of Reductio. While Ferreira concludes that this is a problem for bilateralism as such, del Valle-Inclan and Schlöder (2023) argue that this is just a problem for the specific rules that Rumfitt provides for conjunction and disjunction: rules of the form of BNK1. The BNK2 rules, as del Valle-Inclan and Schlöder show, meet this constraint. Schematizing the proofs they provide for this specific case where BNK1 fails, we have the following reduction:

$$
\begin{array}{c} \overline{\boldsymbol{c}\langle\varphi\circ\psi\rangle}^{\,1} \\ \mathcal{D}_1 \\ \dfrac{\bot}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\text{Red.}^{1} \end{array}
\qquad\rightsquigarrow\qquad
\begin{array}{c} \dfrac{\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1} \quad \overline{\boldsymbol{b}\langle\psi\rangle}^{\,2}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}_{\circ\mathrm{I}} \\ \mathcal{D}_1 \\ \dfrac{\dfrac{\bot}{\boldsymbol{b}^*\langle\psi\rangle}\,\text{Red.}^{2}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{}_{\circ\mathrm{I}}{}^{1} \end{array}
$$

Thus, in any case in which we assume $\boldsymbol{c}\langle\varphi\circ\psi\rangle$ and derive an incoherence to conclude $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$ by way of Reductio, we could just as well assume $\boldsymbol{a}\langle\varphi\rangle$ and $\boldsymbol{b}\langle\psi\rangle$, use Reductio only on $\boldsymbol{b}\langle\psi\rangle$, a formula of lesser complexity, and then conclude $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$ by way of the $\boldsymbol{c}^*$ introduction rule. BNK2, however, is not the only system that meets this constraint: BNK3 meets it just as well. For this specific case, we have the following reduction:

$$
\begin{array}{c} \overline{\boldsymbol{c}\langle\varphi\circ\psi\rangle}^{\,1} \\ \mathcal{D}_1 \\ \dfrac{\bot}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\text{Red.}^{1} \end{array}
\qquad\rightsquigarrow\qquad
\begin{array}{c} \dfrac{\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1} \quad \overline{\boldsymbol{b}\langle\psi\rangle}^{\,2}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}_{\circ\mathrm{I}} \\ \mathcal{D}_1 \\ \dfrac{\bot}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{}_{\circ\mathrm{I}}{}^{1,2} \end{array}
$$

Whereas, in the case of BNK2, we reduce the complexity of the Reductio'd formula, in the case of BNK3 shown here, we eliminate the use of Reductio entirely. Similar reductions can be given for all of the applications of coordination principles on logically complex formulas for BNK3.[11] So, on this second approach to bilateral harmony, BNK2 and BNK3 are bilaterally harmonious, whereas BNK1 is not.

This second approach to bilateral harmony fares much better than Francez's in providing a *general* constraint that can be applied to any set of rules, no matter their form. Unlike Francez's criterion, del Valle-Inclan and Schlöder's approach rules out the BNK3-form *bonk* rules. However, it still has the same basic problem: it can plausibly be regarded as a *sufficient* condition of bilateral harmony, but, conceived of as a *necessary* condition, it is too strong, ruling out intuitively harmonious rules as disharmonious. In this case, it is the BNK1 rules that get the boot. Once again, for a concrete case, consider just the positive and negative conjunction introduction rules of BNK1:

[11] Since the elimination rules of BNK2 and BNK3 are the same, the reductions of Incoherence are the same. The reduction of the other direction of Reductio for BNK3, where $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$ assumed to conclude $\boldsymbol{c}\langle\varphi\circ\psi\rangle$ is similar to that for BNK2 shown by del Vall-Inclan and Schlöder.

$$\frac{+\langle\varphi\rangle \quad +\langle\psi\rangle}{+\langle\varphi \wedge \psi\rangle}\ +_{\wedge\mathrm{I}} \qquad \frac{-\langle\varphi\rangle}{-\langle\varphi \wedge \psi\rangle}\ -_{\wedge\mathrm{I}_1} \qquad \frac{-\langle\psi\rangle}{-\langle\varphi \wedge \psi\rangle}\ -_{\wedge\mathrm{I}_2}$$

Once again, these positive and negative rules clearly *seem* to be harmonious. Indeed, it is just the intuition that "a conjunction [which is assertible upon the assertion of both conjunctions (together)] should be deniable upon the denial of each conjunct (separately)" that leads Francez (2015, 159) to propose his criterion of bilateral harmony which generalizes this intuition. Now, I've just argued that there may be harmonious rules of different forms that fail to meet Francez's specific criterion, and so it does not constitute a *necessary* criterion of bilateral harmony. Nevertheless, it still does seem to be a *sufficient* criterion in that positive and negative rules that meet it are neither too strong nor too weak relative to each other. At the very least, if one is going to claim that rules that meet it are disharmonious, one ought to have a good reason to do so. However, the only explicit motivation that del Valle-Inclan and Schlöder provide for imposing their criterion of harmony on a set of bilateral rules is that it rules out connectives like *bonk* and *conk*, but, as we've seen, requiring that rules conform to Francez's criterion of harmony suffices to rule out connectives like *bonk* and *conk* as well. So, once again, it seems that we are given a criterion of bilateral harmony that is *sufficient* but not *necessary*.[12]

There is one more approach to bilateral harmony in the literature that I cannot go into here for reasons of space, and that is the approach articulated by Kürbis (2021). Kürbis puts forward a normalization procedure for bilateral natural deduction systems with several reduction steps. While the various steps *do* yield an extensionally adequate criterion of bilateral harmony, there is no clear principle of unity among the various conditions, and so the approach itself ends up looking rather ad hoc. This is something that Kürbis himself admits, claiming that, while this approach technically gets the result and rules out connectives like *conk*, there is reason to think that it "does not really go to the heart of the matter of what is wrong with *conk*," (553). The criterion for unilateral harmony systematically rules out *tonk* and *tunk* in a way that is conceptually illuminating, getting to the heart of the matter as to what is wrong with these connectives. We should aspire to a criterion for bilateral harmony that does the same thing. Kürbis's account, self-admittedly, does not.

[12]Now, to be clear, I have no objection to del Valle-Inclan and Schlöder's proposal as a specification of a *desirable* property for a bilateral system to have, providing some grounds (indeed, perhaps even decisive grounds) to prefer either BNK2 or BNK3 over BNK1 in the context of providing a proof-theoretic semantics for classical logic. My claim is just that it should not be regarded as a criterion of bilateral harmony.

5 A New Approach to Bilateral Harmony

I now turn to the task of articulating a new criterion of bilateral harmony. It is not hard to articulate, informally, what is wrong with *bonk* and *conk*. Intuitively, the problem with the positive and negative introduction rules for *bonk* is that it is *too easy* to introduce opposite stances towards φ *bonk* ψ. In particular, introducing opposite stances towards some sentence, which enables one to conclude an incoherence, should require grounds that are themselves incoherent. In the case of *bonk*, however, one can conclude $+\langle\varphi \textit{ bonk } \psi\rangle$ and $-\langle\varphi \textit{ bonk } \psi\rangle$, two opposite stances, from $+\langle\varphi\rangle$ and $-\langle\psi\rangle$, two stances which are not themselves guaranteed to be incoherent. On the other hand, once again very informally, the problem with the positive and negative introduction rules for *conk* is that it is *too hard* to introduce opposite stances towards φ *conk* ψ. In particular, to introduce $+\langle\varphi \textit{ conk } \psi\rangle$ we need to have both $+\langle\varphi\rangle$ and $+\langle\psi\rangle$ and, to introduce $-\langle\varphi \textit{ conk } \psi\rangle$ we need to have both $-\langle\varphi\rangle$ and $-\langle\psi\rangle$. Intuitively, this is too much! The conditions for denying φ *conk* ψ should not just be *incompatible* with the affirmation conditions, but should be *minimally* incompatible. Insofar as we need $-\langle\varphi\rangle$ and $-\langle\psi\rangle$ to affirm φ *conk* ψ, needing *both* $-\langle\varphi\rangle$ and $-\langle\psi\rangle$ to deny φ *conk* ψ when *either* by itself is already incompatible with the conditions for affirming φ *conk* ψ violates this minimality constraint. Our task in formulating a criterion of bilateral harmony, then, is to turn these two informal conditions into a pair of formal constraints.

The formal constraint corresponding to the first informal condition is obvious. We need to be able to show that, in any case in which opposite stances towards $\varphi \circ \psi$ are introduced and an incoherence is concluded on that base, the grounds for introducing these opposite stances already suffice to conclude an incoherence without the introduction of opposite stances towards $\varphi \circ \psi$. This amounts to establishing a reduction procedure. For BNK1, the reduction with the first $\boldsymbol{c}^*$ introduction rule goes as follows:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\boldsymbol{a}\langle\varphi\rangle} \quad \dfrac{\mathcal{D}_2}{\boldsymbol{b}\langle\psi\rangle}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}_{\circ\mathrm{I}} \qquad \dfrac{\dfrac{\mathcal{D}_3}{\boldsymbol{a}^*\langle\varphi\rangle}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{}_{\circ\mathrm{I}_1}}{\perp}\,\text{Inc.} \quad \rightsquigarrow_r \quad \dfrac{\dfrac{\mathcal{D}_1}{\boldsymbol{a}\langle\varphi\rangle} \quad \dfrac{\mathcal{D}_3}{\boldsymbol{a}^*\langle\varphi\rangle}}{\perp}\,\text{Inc.}$$

The reduction with the second $\boldsymbol{c}^*$ introduction rule is analogous. For BNK3, the reduction goes as follows:

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \boldsymbol{a}\langle\varphi\rangle & \boldsymbol{b}\langle\psi\rangle\end{matrix}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}_{\circ}\mathrm{I} \qquad \dfrac{\begin{matrix}\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1} & \overline{\boldsymbol{b}\langle\psi\rangle}^{\,2}\\ \multicolumn{2}{c}{\mathcal{D}_3}\\ \multicolumn{2}{c}{\bot}\end{matrix}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{}_{\circ}\mathrm{I}^{1,2}}{\bot}\,\text{Inc.} \quad \rightsquigarrow_r \qquad \begin{matrix}\mathcal{D}_1 & \mathcal{D}_2\\ \boldsymbol{a}\langle\varphi\rangle & \boldsymbol{b}\langle\psi\rangle\\ \multicolumn{2}{c}{\mathcal{D}_3}\\ \multicolumn{2}{c}{\bot}\end{matrix}$$

The reduction for BNK2 is similar. Note that, in the case of *bonk*, there is a combination of $\boldsymbol{c}$ and $\boldsymbol{c}^*$ introduction rules such that no reduction is possible. This formally captures the problem with *bonk* informally articulated above.

It is less obvious as to how the problem with *conk*, informally articulated above, ought to be formally captured. However, I take it that we can arrive at a satisfactory formal constraint by thinking of bilateral harmony by analogy to unilateral harmony. Just as, in the case of unilateral harmony, a reduction shows that the elimination rules are not too strong relative to the introduction rules, and an expansion shows that they're not too weak relative to the introduction rules, in the case of bilateral harmony, a *reduction* shows that it's not *too easy* to conclude opposite stances towards $\varphi \circ \psi$, an *expansion* can show that it's not *too hard* ton conclude opposite stances towards $\varphi \circ \psi$. In the case of expansions establishing unilateral harmony, we suppose we have a derivation of $\boldsymbol{c}\langle\varphi\circ\psi\rangle$, we then use the $\boldsymbol{c}_\circ$ elimination rules, making whatever assumptions we must make in order to use them, and derive the grounds required to apply the $\boldsymbol{c}_\circ$ introduction rules and recover $\boldsymbol{c}\langle\varphi\circ\psi\rangle$, having discharged all of our assumptions. Extending this thought analogically, we can think of the application of Incoherence and Reductio, given $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$, as the application of a kind of elimination rule.[13] This suggests the following expansion procedure. We suppose we have a derivation of $\boldsymbol{c}\langle\varphi\circ\psi\rangle$, and we make whatever assumptions necessary in order to apply the introduction rule for $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$. Then, using the coordination principles and the introduction rules for $\boldsymbol{c}\langle\varphi\circ\psi\rangle$, we must be able to recover that formula, having discharged all of our assumptions. The fact that we *can* discharge all of our assumptions and reintroduce $\boldsymbol{c}\langle\varphi\circ\psi\rangle$, means that we didn't have to assume *too much* to conclude $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$. That is, it's not "too hard" to conclude $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$, relative to how hard it is to introduce $\boldsymbol{c}\langle\varphi\circ\psi\rangle$. Likewise, we suppose we have a derivation of $\boldsymbol{c}^*\langle\varphi\circ\psi\rangle$ and do the same procedure.

The expansions for BNK1 go as follows:

[13] Of course, coordination principles can also be likened to introduction rules (cf. Kürbis (2021)). The reason for thinking of likening coordination principles to elimination rules here is simply that, following Gentzen's principle of prioritizing introduction rules in the context of proof-theoretic semantics approach, the criterion of bilateral harmony is formulated for positive and negative introduction rules.

$$\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array} \rightsquigarrow_e \dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array} \quad \dfrac{\overline{\boldsymbol{a}^*\langle\varphi\rangle}^{\,1}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{\circ}\mathrm{I}_1}{\bot}\,\mathrm{Inc.}}{\boldsymbol{a}\langle\varphi\rangle}\,\mathrm{Red.}^1 \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array} \quad \dfrac{\overline{\boldsymbol{b}^*\langle\psi\rangle}^{\,2}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{\circ}\mathrm{I}_2}{\bot}\,\mathrm{Inc.}}{\boldsymbol{b}\langle\psi\rangle}\,\mathrm{Red.}^2}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}{\circ}\mathrm{I}$$

$$\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}^*\langle\varphi\circ\psi\rangle\end{array} \rightsquigarrow_e \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}^*\langle\varphi\circ\psi\rangle\end{array} \quad \dfrac{\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1} \quad \overline{\boldsymbol{b}\langle\psi\rangle}^{\,2}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}{\circ}\mathrm{I}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{a}^*\langle\varphi\rangle}\,\mathrm{Red.}^1}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{\circ}\mathrm{I}_1 \quad \overline{\boldsymbol{c}\langle\varphi\circ\psi\rangle}^{\,3}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{b}^*\langle\psi\rangle}\,\mathrm{Red.}^2}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{\circ}\mathrm{I}_2 \quad \overline{\boldsymbol{c}\langle\varphi\circ\psi\rangle}^{\,3}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\mathrm{Red.}^3$$

The expansions for BNK3 go as follows:

$$\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array} \rightsquigarrow_e \dfrac{\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array} \quad \dfrac{\dfrac{\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1} \quad \overline{\boldsymbol{a}^*\langle\varphi\rangle}^{\,2}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{\circ}\mathrm{I}^{1,0}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{a}\langle\varphi\rangle}\,\mathrm{Red.}^2 \qquad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}\langle\varphi\circ\psi\rangle\end{array} \quad \dfrac{\dfrac{\overline{\boldsymbol{b}\langle\psi\rangle}^{\,3} \quad \overline{\boldsymbol{b}^*\langle\psi\rangle}^{\,4}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{\circ}\mathrm{I}^{3,0}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{b}\langle\psi\rangle}\,\mathrm{Red.}^4}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}{\circ}\mathrm{I}$$

$$\begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}^*\langle\varphi\circ\psi\rangle\end{array} \rightsquigarrow_e \dfrac{\dfrac{\dfrac{\overline{\boldsymbol{a}\langle\varphi\rangle}^{\,1} \quad \overline{\boldsymbol{b}\langle\psi\rangle}^{\,2}}{\boldsymbol{c}\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}{\circ}\mathrm{I} \quad \begin{array}{c}\mathcal{D}_1\\ \boldsymbol{c}^*\langle\varphi\circ\psi\rangle\end{array}}{\bot}\,\mathrm{Inc.}}{\boldsymbol{c}^*\langle\varphi\circ\psi\rangle}\,\boldsymbol{c}^*{\circ}\mathrm{I}^{1,2}$$

The expansions for BNK2 are similar. No such expansions are possible in the case of *conk*.

I submit that this criterion of bilateral harmony provides both necessary and sufficient conditions for a set of positive and negative rules being harmonious. On the one hand, it is satisfied by BNK1, BNK2, and BNK3, three systems whose rules are intuitively harmonious. On the other hand, it rules out *bonk*, *conk*, and every other bilaterally dissonant connective that has been proposed in the literature. Moreover, it really does get to the heart of the matter as to what is wrong with these connectives.

6 Conclusion

In this paper, I've done three main things: (1) I've presented, in generalized fashion, three bilateral natural deduction systems for classical logic, (2) I've provided a new criterion for bilateral harmony that I argued is superior to existing criteria that have been proposed in the literature, and (3) I've shown that all three systems meet it. I'll briefly conclude with three directions for further work.

First, I have argued that it is a virtue of my proposed criterion of bilateral harmony that all three intuitively harmonious systems meet it. A consequence of this, however, is that a bilateral natural deduction system's being both unilaterally and bilaterally harmonious does not suffice to establish that its rules are uniquely definitive of the meanings of the classical connectives, since there are multiple such sets of rules. So, one has a choice: either supplement harmony with some further proof-theoretic constraint to distinguish one set of rules as uniquely definitive or give up on the idea that there is some one such set of rules. If one opts for the first option, then one such constraint, giving some grounds to prefer BNK2 or BNK3 over BNK1, may be that proposed by del Valle-Inclan and Schlöder. However, some further grounds will be needed to decide between BNK2 and BNK3. If one goes in for the second option, then one needs to say, what, exactly, a proof-theoretic specification of the meanings of the classical connectives comes to if not the specification of a set of rules that define the meanings of the connectives. Either way, there is work to be done.

Second, though my formulation of bilateral harmony is novel in the context of bilateral natural deduction systems, as is well-known, there is a very close correspondence between bilateral systems and multiple conclusion sequent calculi. While there is not the space here to develop this claim, my two constraints correspond very closely to two constraints often placed on multiple conclusion sequent calculi in the context of proof-theoretic semantics (Hacking (1979), Kremer (1988)): the eliminability of Cut and the eliminability of non-atomic instances of the Identity axiom. I've developed this same basic conception of bilateral harmony, under this different guise, elsewhere. A developed account of the relation between these two approaches would be illuminating.

Finally, I have restricted my attention here to *classical* logic, as this has been the main focus of developments in bilateralism following Smiley and Rumfitt. However, there has been developments of bilateral natural deduction for non-classical logics in recent years.[14] This general schematized approach to bilateralism as well as the more specific approach to bilateral harmony may be fruitfully applied to such developments. In this regard, it is important that the criterion of bilateral harmony proposed here is a more permissive one than those that have been proposed in the literature, since, while the BNK2 and BNK3 rules are suitable for classical logic, they will not be suitable for many non-classical logics. So, though my focus here has been classical logic, the menu of harmonious bilateral systems given in this paper, the generalized approach through which they've been stated, and the method for establishing bilateral harmony will likely be of use to those looking to apply bilateralism beyond classical logic.

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[14] See note 2 above for some such developments. See also Francez (2014b) and Francez (2023).

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