# 3

# Sequent Systems

## 3.1 Introduction

In the last chapter, we considered natural deduction, a kind of proof system introduced in Gerhard Gentzen's 1935 dissertation. However, in this dissertation, Gentzen actually introduced *two* new kinds of proof systems. The first was natural deduction, and the second kind of system is called the *sequent calculus*. The key difference between these two systems is that, whereas natural deduction systems provide rules for moving from *particular sentences* to other particular sentences, sequent systems provide rules for moving from *sequents*, which themselves encode inferential relations between particular sentences, to other sequents. So, whereas one operates at the *first-order inferential* level in using a natural deduction system, one operates at the *meta-inferential* level in using a sequent calculus. In this chapter, we'll consider using the sequent calculus as a way of providing an inferentialist account of the meanings of the logical connectives belonnging to classical logic.

## 3.2 A Single Conclusion Sequent Calculus

Natural deduction systems provide rules for inferring from sentences to other sentences. For instance, the conjunction elimination rule lets us infer from $A \wedge B$ to $A$. However, natural deduction systems also provide rules that are, in effect, rules for inferring sentences from inferential relations between sentences. For instance, the conditional introduction rule lets us infer the sentence $A \rightarrow B$ from the proof of $B$ under the assumption of $A$. So, in a sense, we infer $A \rightarrow B$ from the inferential relation that obtains between $A$ and $B$. Now, when we prove $B$ under the assumption of $A$, we can use all of our premises $X$, and, likewise,

when we conclude $A \to B$, we are concluding it given our premises $X$. So, the conditional introduction rule might be understood as saying that if, given our premises $X$ and the additional premise of $A$, we can conclude $B$, then, from our premises $X$, we can conclude $A \to B$. We might put this rule as follows:

$$\frac{X, A \vdash B}{X \vdash A \to B} \; R_{\to}$$

This is a sequent rule. In a sequent calculus, the basic items that the system provides rules for inferring between are not sentences, but *sequents*, which themselves encode inferential relations between sentences. A sequent of the form $X \vdash A$, where $X$ is a set of sentences and $A$ is a single sentence, can be understood as saying that the conclusion $A$ follows from the premises $X$, that $X$ implies $A$.[1] A sequent calculus is a proof system with rules for deriving such sequents from other such sequents.

Unlike a natural deduction system, where the connectives are given introduction and elimination rules, in the sequent calculus, logical connectives are given just introduction rules: a rule (or number of rules) for introducing a sentence with that connective on the *right* side of a sequent, and a rule (or number of rules) for introducing a sentence with that connective on the *left* side of a sequent. We've just given the right rule for the conditional. Consider now the left rule:

$$\frac{X \vdash A \quad X, B \vdash C}{X, A \to B \vdash C} \; \mathrm{L}_{\to}$$

This says that, if $X$ implies $A$ and $X$ along with $B$ implies $C$, then $X$ along with $A \to B$ implies $C$. Intuitively, this specifies just the conditions under which a set of sentences $X$ along with $A \to B$ implies some sentence $C$. In general, whereas the right rule for a binary connective $\circ$ codifies the inferential relations that a set of sentences $X$ must bear to $A$ and $B$ in order for it to be the case that $X$ implies $A \circ B$, the left rule codifies the inferential relations that $X$ must bear to $A$, $B$, and

[1] Note, in linguistics "implication" is often taken to denote a specifically pragmatic phenomenon, distinct from semantic entailment. For instance, whereas "I ate some of the ice cream" *semantically entails* "I ate some food" it *pragmatically implies* "I didn't eat all of the ice cream." When "implication" is used in this book, it is *not* being used in this technical sense—it is, rather, just a term used to express the intuitive notion of something's following from something else.

$C$, in order for it to be the case that $X$ along with $A \circ B$ implies $C$. Thus, the right rules can be understood codifying the conditions required for $A \circ B$ to figure as a *conclusion* of a good inference, and the left rules can be understood as codify the conditions required for $A \circ B$ to figure as a *premise* of a good inference. In this way, providing right and left rules for a logical connective in the context of a sequent calculus can be understood as codifying the role of a sentence containing that connective as both conclusion and premise of inferences.

Having stated the rules for the conditional, let us turn now to the rules for conjunction and disjunction. Consider first the conjunction rules:

$$\frac{X \vdash A \quad X \vdash B}{X \vdash A \wedge B}\ \mathrm{R}_{\wedge} \qquad\qquad \frac{X, A, B \vdash C}{X, A \wedge B \vdash C}\ \mathrm{L}_{\wedge}$$

The right rule says that if some set of sentences $X$ implies $A$ and $X$ also implies $B$, then $X$ implies $A \wedge B$. Once again, this right rule clearly corresponds to the introduction rule of the natural deduction system. The left rule says that if some conclusion $C$ follows from $X$ along with $A$, along with $B$, then $C$ follows from $X$ along with $A \wedge B$. Quite intuitive. Consider now the disjunction rules:

$$\frac{X \vdash A}{X \vdash A \vee B}\ \mathrm{R}_{\vee_1} \qquad \frac{X \vdash B}{X \vdash A \vee B}\ \mathrm{R}_{\vee_2} \qquad \frac{X, A \vdash C \quad X, B \vdash C}{X, A \vee B \vdash C}\ \mathrm{L}_{\vee}$$

Here, we have two right rules, once again corresponding to our two natural deduction introduction rules. They tell us that if $X$ implies $A$, then $X$ implies $A \vee B$, and, likewise, if $X$ implies $B$, then $X$ implies $A \vee B$. The left rule, which, in this case, corresponds directly to the elimination rule of our sequent calculus, telling us.

As before, the proofs in a sequent system take the form of *trees*. In order to simply prove that some sentence follows from some other set of sentences, we need *axioms* that we can. Consider again *reflexivity*, the principle that $A$ follows from $A$. In the natural deduction system put forward in the previous chapter, this was built into the structure of the system. In a sequent calculus, we have stipulate it as a rule:

$$\frac{}{A \vdash A}\ \text{Reflex.}$$

You can think of this as a *zero-premise* rule. That is, you don't need any premises to conclude a sequent of the form $A \vdash A$—you can always conclude it, using a sequent of this form a "leaf" of your proof tree. This rule contrasts with the conjunction rules stated above in that it doesn't concern the use of any logical operators. Rules that concern the use of logical operators are called *operational rules*, whereas rules like this, which concern the structure of the "follows from" relation codified by $\vdash$, are called *structural rules*. Consider also the principle of *monotonicity*, which says that if some sentence $A$ follows from some set of sentences $X$, then $A$ follows from $X$ along with some other sentence $B$. Once again, whereas, in our natural deduction system, this rule was built into the structure of the proof system itself, in a sequent calculus we have to stipulate it as a structural rule:

$$\frac{X \vdash A}{X, B \vdash A}\ \text{Mono.}$$

What about transitivity, the principle that if $A$ follows from $X$, and $B$ follows from $Y$, then $B$ follows from $X$ along with $Y$? Surely, we want this to hold as well, and so we may well stipulate it as an explicit structural rule:

$$\frac{X \vdash A \quad Y, A \vdash B}{X, Y \vdash B}\ \text{Cut}$$

Though this is just the transitivity rule we've already considered, I've used the more famous name given to this rule by Gentzen: *Cut*. The rule is called "Cut" since it states that when we have a proof from $X$ to $A$ and from $Y$ along with $A$ to $B$, we can "cut out the middle man," as it were, and go directly from $X$ along with $Y$ to $B$.

To see a simple proof in this system, using these structural rules and the rules we've just given for the conditional and conjunction, consider the proof of $p \to q \vdash (p \wedge r) \to q$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\ \text{Reflex.}}{p, r \vdash p}\ \text{Mo.} \quad \dfrac{\dfrac{}{q \vdash q}\ \text{Reflex.}}{p, r, q \vdash q}\ \text{Mo.}}{p \to q, p, r \vdash q}\ L_{\to}}{p \to q, p \wedge r \vdash q}\ L_{\wedge}}{p \to q \vdash (p \wedge r) \to q}\ R_{\to}$$

For a slightly more complicated proof, consider the proof of $(p \wedge r) \vee (q \wedge s) \vdash (p \vee q) \wedge (r \vee s)$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Reflex.}}{p, r \vdash p}\,\text{Mo.}}{p, r \vdash p \vee q}\,R_{\vee_1} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{r \vdash r}\,\text{Reflex.}}{p, r \vdash r}\,\text{Mo.}}{p, r \vdash r \vee s}\,R_{\vee_1}}{p, r \vdash (p \vee q) \wedge (r \vee s)}\,R_{\wedge}}{p \wedge r \vdash (p \vee q) \wedge (r \vee s)}\,L_{\wedge} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{q \vdash q}\,\text{Reflex.}}{q, s \vdash q}\,\text{Mo.}}{q, s \vdash p \vee q}\,R_{\vee_2} \quad \dfrac{\dfrac{\dfrac{\dfrac{}{s \vdash s}\,\text{Reflex.}}{q, s \vdash s}\,\text{Mo.}}{q, s \vdash r \vee s}\,R_{\vee_2}}{q, s \vdash (p \vee q) \wedge (r \vee s)}\,R_{\wedge}}{q \wedge s \vdash (p \vee q) \wedge (r \vee s)}\,L_{\wedge}}{(p \wedge r) \vee (q \wedge s) \vdash (p \vee q) \wedge (r \vee s)}\,L_{\vee}$$

These proofs might not look as intuitive as the natural deduction proofs given in the previous chapter, but they do make intuitive sense if you think through them, and they are in fact quite easy to construct. In general, when constructing proofs using the sequent calculus, it is easiest to work *backwards*, starting with the sequent you want to prove at the trunk of the tree and using the rules to deconstruct the sentences in this sequent up the tree into sequents of the form $X, A \vdash A$, which can be derived via several applications of Monotonicity (as many as there are elements in $X$) and Reflexivity.

Now, though it is generally easiest to work backwards in constructing a proof in a sequent calculus, in this sequent calculus, when working backwards from a provable sequent, you will still have to make some choices about which rules to use to arrive at axioms. To take a trivial example, consider the proof of $p \vdash p \vee q$. If you work backwards from this sequent, applying the rules in reverse to deconstruct the logically complex sentences in this sequent to eventually arrive at axioms (in this case, there is only one such sentence), you have to be sure to apply the *first* right disjunction rule, so as to derive the axiom $p \vdash p$. If, instead, you applied the second right disjunction, you'll end up with $p \vdash q$, and you'll have no proof. Of course, in this case, it is obvious which rule to apply to end up where you want to go. However, in more complicated proofs, it may not be so obvious which rules to apply at which point. Later in this chapter, we will introduce a variant of this sequent calculus that takes all of the choice out of it, where, working backwards, you can blindly apply any rules that you can and you'll be sure to arrive at axioms just in case the sequent is provable.

Having introduced operational rules for the binary connectives and the structural rules of this system, let us finally turn to the rules for negation. There are two different ways to formulate the negation rules. One way to put the rules is with the use of the $\bot$ symbol we introduced in the previous chapter:

$$\frac{X, A \vdash \bot}{X \vdash \neg A}\ \mathrm{R}_\neg \qquad\qquad \frac{X \vdash A}{X, \neg A \vdash \bot}\ \mathrm{L}_\neg$$

The right rule, once again, is just the introduction rule from our natural deduction system. It says that if $X$ along with $A$ implies a contradiction, then $X$ implies $\neg A$. The left rule, on the other hand, says that if $X$ implies $A$, then $X$ along with $\neg A$ implies a contradiction. Though this is how the rules for negation are usually formulated, in fact, we don't really need to use the symbol $\bot$ at all. Instead, we can simply think of a sequent of the form $X \vdash$, with an empty right-hand side, as saying that the sentences in $X$ are jointly inconsistent. So, we can write the rules without the $\bot$ symbol simply as follows:

$$\frac{X \vdash A}{X, \neg A \vdash}\ \mathrm{L}_\neg \qquad\qquad \frac{X, A \vdash}{X \vdash \neg A}\ \mathrm{R}_\neg$$

Monotonicity then can be understood as a structural rule pertaining to *both* sides of a sequent. That is, we have the *Left* Monotonicity rule stated above, but also a *Right* Monotonicity rule:

$$\frac{X \vdash A}{X, B \vdash A}\ \text{L. Mono.} \qquad\qquad \frac{X \vdash}{X \vdash A}\ \text{R. Mono.}$$

Thus, for instance, we can prove $p \wedge \neg p \vdash q$ as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\ \text{Reflex.}}{p, \neg p \vdash}\ \mathrm{L}_\neg}{p \wedge \neg p \vdash}\ \mathrm{L}_\wedge}{p \wedge \neg p \vdash q}\ \text{Mono.}$$

We can read this proof as follows: $p$ implies $p$. So, $p$ along with $\neg p$ is inconsistent. Accordingly, $p \wedge \neg p$ is inconsistent. Thus, $p \wedge \neg p$ implies $q$. The principle of Explosion is, in this way, built in as a structural feature of our system.

We've now provided a sequent calculus in which each connective is given right and left rules. Though logical inferentialists often opt for natural deduction in formulating their logical inferentialism, it's clear that a sequent calculus such as this one can just as well serve as the setting for formulating logical inferentialism. That is, right and left rules in a sequent calculus can serve just as well to define the meanings of logical connectives as introduction and elimination rules in a natural deduction system. Once again, we can think of the inferential role of a sentence in terms of two aspects: its role as a *conclusion* of good inferences and its role as a *premise* of good inferences. We've seen that the *right* rules for a logical connective codify the conditions for a sentence with that connective as its main operator to serve as the *conclusion* of a good inference and the *left* rules codify the conditions for such a sentence serve as the *premise* of a good inference. Thus, one can maintain that the rules of the sequent calculus serve as inferentialist "definitions" of the meaning of a connective.

## 3.3 Tonk and Tunk Again

At this point, we have a familiar problem that must be dealt with. Once again, one might be tempted to think any connective defined by way of right and left rules in a sequent calculus can be understood as having a legitimate meaning. However, by now, we should know better. Consider again the connective **tonk**, but this time given rules in a sequent setting:

$$\frac{X \vdash A}{X \vdash A \textbf{ tonk } B}\ \mathrm{R}_{\textbf{tonk}_1} \qquad \frac{X \vdash B}{X \vdash A \textbf{ tonk } B}\ \mathrm{R}_{\textbf{tonk}_2} \qquad \frac{X, A, B \vdash C}{X, A \textbf{ tonk } B \vdash C}\ \mathrm{L}_{\textbf{tonk}}$$

We saw, in the natural deduction context, **tonk** had the *introduction* rules of disjunction, and the *elimination* rules of conjunction. Here, in the context of a sequent calculus, **tonk** has the *right* rules of disjunction, and the *left* rule of conjunction. Once again, defining **tonk** in this way and adding it to our language leads to triviality. Given these operational rules and our structural rules, we can derive anything from anything else as follows:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Reflex.}}{p \vdash p \textbf{ tonk } q}\,\mathrm{R}_{\textbf{tonk}_1} \qquad \dfrac{\dfrac{\dfrac{}{q \vdash q}\,\text{Reflex.}}{p, q \vdash q}\,\text{Mo.}}{p \textbf{ tonk } q \vdash q}\,\mathrm{L}_{\textbf{tonk}}}{p \vdash q}\,\text{Cut}$$

So, given the right rules, one can conclude from the fact that $p$ implies $p$ that $p$ implies $p$ **tonk** $q$, and, given the left rules, one can conclude from the fact that $p$ along with $q$ implies $q$ that $p$ **tonk** $q$ implies $q$. Thus, given the transitivity of the consequence relation, one can conclude, for any sentences $p$ and $q$, that $p$ implies $q$. Just as introducing **tonk** in the previous chapter trivialized out natural deduction system, enabling us to prove anything from anything else, **tonk** here trivializes our sequent calculus.

Once again, it's not too hard to see, intuitively, what's wrong with **tonk**, as defined by these right and left rules. Given the right rules, what a set of sentences must imply in order to imply $A$ **tonk** $B$ is *either $A$ or $B$*. However, given the left rule, $A$ **tonk** $B$ implies anything that *$A$ along with $B$* implies. In other words, when *concluding* **tonk** from other sentences, it functions like *disjunction*, but, when using **tonk** as a *premise* to conclude other sentences, it functions like *conjunction*. That is, of course, just the inferential behavior of **tonk**, as we've already seen in a natural deduction context. In this sequent calculus, however, we will need to formulate a new official criterion to rule out this connective.

Before we turn to task of formulating a harmony criterion that will rule out **tonk**, let us consider again **tunk**, which is the opposite of **tonk**. In a sequent calculus, **tunk** may be defined by the following rules:

$$\dfrac{X \vdash A \quad X \vdash B}{X \vdash A \textbf{ tunk } B}\,\mathrm{R}_{\textbf{tunk}} \qquad\qquad \dfrac{X, A \vdash C \quad X, B \vdash C}{X, A \textbf{ tunk } B \vdash C}\,\mathrm{L}_{\textbf{tunk}}$$

So, whereas **tonk** has the right rules of disjunction and the left rule of conjunction, **tunk** has the right rule of *conjunction* and the left rule of *disjunction*. Thus, when *concluding* **tunk** from other sentences, it functions like *conjunction*, but, when using **tunk** as a *premise* to conclude other sentences, it functions like *disjunction*. As before with natural deduction, introducing **tunk** into our language does not trivialize it as **tonk** does. Still, these rules are intuitively unharmonious in a way that is analogous to **tonk**, and so a criterion of harmony

for the the sequent calculus ought to rule them out. Let us now turn to the task of formulating such a criterion.

## 3.4 Harmony: Cut and Reflexivity Complexity Reductions

Consider again the principle of Cut, which we have included as a structural rule of our sequent calculus:

$$\frac{X \vdash A \quad Y, A \vdash B}{X, Y \vdash B} \text{ Cut}$$

Though we have included Cut as a rule of the sequent calculus, in Gentzen's dissertation, the "main theorem" he proves (the "Hauptsatz" in German) is that this structural is actually *eliminable* from the sequent calculus. That is, while we *can* stipulate Cut as a structural rule and use it in proofs, we never actually *have to* use this structural rule; anything we can prove with Cut, we can just as well prove without it. I won't go through the full Cut Elimination proof here, since doing so is really not essentially for our purposes, but I will sketch the basic strategy of the proof.

We'll call the sentence in the place of $A$ in the above schema, which occurs on the right and left of the two premise sequents and gets eliminated through an application of Cut, the *Cut sentence*. The proof of Cut elimination is a *double induction*, with a *primary* induction on the *complexity* of the Cut sentence with a *secondary* induction on the *height* of the proofs in which the Cut rule is applied.[2] So, the main aim is to show by induction that Cut on formulas of any complexity is eliminable. To establish the *base case* of this *main* induction, one must show that Cut on *atomic* sentences is eliminable, and the way one shows *this* is by *another* induction. In this secondary induction, which is an induction on the combined height of the branches of the proof tree above the application of Cut, one shows, first, that Cut on atomic sentences with where both premises are axioms will be eliminable (i.e. the result will be another axiom), and then shows that in any case in which one might Cut on atomic sentences after applying some rule to derive some other sentence on some side of the turnstile, one can always push the application of Cut up the proof tree,

[2] If the notion of mathematical induction is not familiar to you, see Appendix B of this chapter.

applying the rule after the application of Cut, and, in this way, reduce the height of the application of Cut. This establishes that Cut on atomic sentences is eliminable, and one then shows that, whenever one has complex sentences, one can push the application of Cut up to the point where the complex sentence was just introduced on both the left and the right side of the two sequents, and, in *this* case, one can always reduce the complexity of the Cut sentence by using Cut on the premises needed to derive these sequents.

If the above explanation of the Cut Elimination proof was not completely clear, that's not too important. Several textbooks go through it in detail.[3] What we'll consider here is the crucial case of the Cut Elimination proof in which one shows that, in the case where one applies the Cut rule on a complex sentence that has just been derived on the lefthand side of one sequent and the righthand side of another, one can reduce the complexity of the Cut sentence by cutting on the premises needed to derive these two sequents. For the case of conjunction the reduction goes as follows:[4]

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{X \vdash A} \quad \dfrac{\mathcal{D}_2}{X \vdash B}}{X \vdash A \wedge B}\,\mathrm{R}_\wedge \quad \dfrac{\dfrac{\mathcal{D}_2}{Y, A, B \vdash C}}{Y, A \wedge B \vdash C}\,\mathrm{L}_\wedge}{X, Y \vdash C}\,\mathrm{Cut}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\mathcal{D}_2}{X \vdash B} \quad \dfrac{\dfrac{\mathcal{D}_1}{X \vdash A} \quad \dfrac{\mathcal{D}_3}{Y, A, B \vdash C}}{X, Y, B \vdash C}\,\mathrm{Cut}}{X, Y \vdash C}\,\mathrm{Cut}$$

In the first proof $A \wedge B$ has been introduced on the right side of a sequent, as a *conclusion* from some set of premises $X$, by way of the right rules and it has been introduced on the left side of a sequent, as a *premise* along with some set of axillary premises $Y$ for some conclusion $C$, by way of the left rules. The Cut rule is then used to "cut out the middle man," $A \wedge B$, and conclude the sequent

[3] See for instance

[4] Note, in the second proof, when we Cut at the final step, we'll technically have multiple copies of the sentences in $X$, which we'd have to eliminate with the structural rule of Contraction to arrive at $X, Y \vdash C$. I discuss this structural rule in the next section. In this section, I suppress the applications of this rule for the sake of simplicity, as it is not relevant to the main point.

that says that $C$ follows from $X$ along with $Y$. In the second proof, one shows that, given the conditions on $X$ and $Y$ needed to conclude $A \wedge B$ as following from $X$ and $C$ following from $A \wedge B$ along with $Y$, one can already conclude that $C$ follows from $X$ along with $Y$, without the detour through introducing $A \wedge B$ as a conclusion from $X$ and as a premise along with $Y$ for the conclusion $C$. So, anything we can derive from $A \wedge B$, given what we need in order for it to function as a premise of an implication, is something we can already derive given what we need derive it as a conclusion of a good implication. This reduction can thus be taken to show that the *consequences* of $A \wedge B$, given the specification of its role as a premise by the left rule, are not too strong relative to the *grounds* of $A \wedge B$, given the specification of its role as a conclusion by the right rule.

Though reductions of this sort figure as a crucial step in the full Cut Elimination proof, they can also be understood as themselves supplying a harmony constraint analogous to the reductions in the context of natural deduction.[5] Concretely, the procedure for reductions of this sort is the following:

**Cut Complexity Reduction Procedure:**

> **Step 1:** Assume you have derivations of whatever premises you need in order to conclude some sentence on the right side of a sequent, using the right rules, and also on the left side of a sequent, using the left rules, so as to eliminate it using Cut.
>
> **Step 2:** Show that you can apply the Cut rule on these premises to prove the ultimately derived sequent *directly*, without the detour through introducing a formula with the connective on the left and the right of the sequent.
>
> **Step 3:** Repeat for each possible combination of right and left rules.

[5] It's worth noting that the Cut Elimination proof for some connective ensures that adding that connective to the language will be a *conservative extension* of that language, since Cut is the only rule that can actually eliminate a sentence and, thus, can take you from sequents containing the new vocabulary to one containing only old vocabulary. Some philosophers, such as Dummett, have suggested that conservativity itself be regarded as a criterion of harmony. However, here we will take it to be the local reductions.

Having already shown the reduction for conjunction, consider now the reductions for disjunction:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{X \vdash A}}{X \vdash A \vee B}\, R_{\vee_1} \quad \dfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{Y, A \vdash C \quad Y, B \vdash C}\, L_{\vee}}{X, Y \vdash C}\, \text{Cut} \qquad \dfrac{\dfrac{\dfrac{\mathcal{D}_1}{X \vdash B}}{X \vdash A \vee B}\, R_{\vee_2} \quad \dfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{Y, A \vdash C \quad Y, B \vdash C}\, L_{\vee}}{X, Y \vdash C}\, \text{Cut}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\dfrac{\mathcal{D}_1 \quad \mathcal{D}_2}{X \vdash A \quad Y, A \vdash C}\, \text{Cut} \qquad\qquad \dfrac{\mathcal{D}_1 \quad \mathcal{D}_2}{X \vdash B \quad Y, B \vdash C}\, \text{Cut}$$

$$\dfrac{\dfrac{\mathcal{D}_1}{X, A \vdash B}}{X \vdash A \to B}\, \mathrm{R}_{\to} \quad \dfrac{\mathcal{D}_2 \quad \mathcal{D}_3}{Y \vdash A \quad Y, B \vdash C}\, \mathrm{L}_{\to}$$

$$\dfrac{}{X, Y \vdash C}\, \text{Cut}$$

$$\Downarrow$$

$$\dfrac{\dfrac{\mathcal{D}_1 \quad \mathcal{D}_2}{X, A \vdash B \quad Y \vdash A}\, \text{Cut} \qquad \mathcal{D}_3}{\dfrac{X, Y \vdash B \qquad Y, B \vdash C}{X, Y \vdash C}\, \text{Cut}}\, \text{Cut}$$

Finally, the reduction for negation is quite simple, going as follows:[6]

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{X, A \vdash}}{X \vdash \neg A}\, R_{\neg} \quad \dfrac{\mathcal{D}_2}{Y \vdash A}}{X, Y \vdash}\, \text{Cut}$$

$$\Downarrow$$

$$\dfrac{\mathcal{D}_2 \quad \mathcal{D}_1}{Y \vdash A \quad X, A \vdash}\, \text{Cut}$$

[6]Note that we allow the conclusion in the Cut rule to be empty.

So, in each case where we can derive a complex sentence on both sides of the turnstile and use Cut to eliminate it, we could just as well use Cut on the premise sequents to derive the same sequent, without the detour through introducing the logical connective on both the left and the right.

Now, consider again the proof that, given the rules for **tonk**, one can conclude $p \vdash q$ for an arbitrary $p$ and $q$:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Reflex.}}{p \vdash p \textbf{ tonk } q}\,\mathrm{R}_{\textbf{tonk}_1} \qquad \dfrac{\dfrac{\dfrac{}{q \vdash q}\,\text{Reflex.}}{p, q \vdash q}\,\text{Mo.}}{p \textbf{ tonk } q \vdash q}\,L_{\textbf{tonk}}}{p \vdash q}\,\text{Cut}$$

Clearly, there is no way to Cut on $p \vdash p$ and $p, q \vdash q$, the premises needed to respectively derive $p \vdash p$ **tonk** $q$ and $p$ **tonk** $q \vdash q$, in order to derive $p \vdash q$. Cutting on these premises merely yields $p, q \vdash q$ again. Officially, the disharmoniousness of the rules for **tonk** is shown by the fact that no reduction of the following sort is possible:

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{X \vdash A}}{X \vdash A \textbf{ tonk } B}\,\mathrm{R}_{\textbf{tonk}_1} \qquad \dfrac{\dfrac{\mathcal{D}_2}{Y, A, B \vdash C}}{Y, A \textbf{ tonk } B \vdash C}\,L_{\textbf{tonk}}}{X, Y \vdash C}\,\text{Cut}$$

$$\Big\downarrow$$

???

Here, there is no way to Cut on the premises $X \vdash A$ and $Y, A, B \vdash C$ to derive $X, Y \vdash C$. Cutting on these premises merely yields $X, Y, B \vdash C$, and there is no way to get rid of the $B$ so as to derive $X, Y \vdash C$. The fact that no reduction in the complexity of the Cut formula here shows that the role of $A$ **tonk** $B$ as a premise is *too strong* relative to its role as a conclusion.

Requiring reductions on Cut formula complexity takes care of **tonk**. What, then, about it's opposite, **tunk**? To see what's wrong with this connective, let us turn our attention to a different structural rule, Reflexivity:

$$\dfrac{}{A \vdash A}\,\text{Reflex.}$$

Here, we allow $A$ to be any sentence, atomic or complex. Thus, not only may we start a proof tree with $p \vdash p$, but, for instance, we may start a proof tree with $p \wedge q \vdash p \wedge q$. But do we *have to* be so permitted? Intuitively, it seems that our the rules for conjunction should enable us to *prove* $p \wedge q \vdash p \wedge q$ from axioms that don't involve conjunction. Indeed, they can, as the following proof shows:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Reflex.}}{p, q \vdash p}\,\text{Mo.} \quad \dfrac{\dfrac{}{q \vdash q}\,\text{Reflex.}}{p, q \vdash q}\,\text{Mo.}}{p, q \vdash p \wedge q}\,R_\wedge}{p \wedge q \vdash p \wedge q}\,L_\wedge$$

In general, where $A$ and $B$ are potentially complex sentences, we can always derive an instance of Reflexivity of the form $A \wedge B \vdash A \wedge B$ from instances of Reflexivity pertaining to the simpler sentences $A$ and $B$. The following transformation shows this:

$$\dfrac{}{A \wedge B \vdash A \wedge B}\,\text{Reflex.} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}}{A, B \vdash A}\,\text{Mo.} \quad \dfrac{\dfrac{}{B \vdash B}\,\text{Reflex.}}{A, B \vdash B}\,\text{Mo.}}{A, B \vdash A \wedge B}\,R_\wedge}{A \wedge B \vdash A \wedge B}\,L_\wedge$$

Intuitively, you can think of this reduction as showing *why* $A \wedge B$ follows from $A \wedge B$ in the context of our system. Given the structural rules of our system, $A$ follows from $A$ along with $B$ and $B$ also follows from $A$ along with $B$. Accordingly, given the right conjunction rule, $A \wedge B$ follows from $A$ along with $B$, and thus, given the left conjunction rule, $A \wedge B$ follows from $A \wedge B$. This can be understood as establishing that the inferential role of $A \wedge B$ as a *premise* is not *too weak* relative to the inferential role of $A \wedge B$ as a *conclusion*.

We can provide reductions of this sort for all of the connective rules we've given. The reduction for disjunction goes as follows:

$$\dfrac{}{A \vee B \vdash A \vee B}\,\text{Reflex.} \qquad \rightsquigarrow \qquad \dfrac{\dfrac{\dfrac{}{A \vdash A}\,\text{Reflex.}}{A \vdash A \vee B}\,R_{\vee_1} \quad \dfrac{\dfrac{}{B \vdash B}\,\text{Reflex.}}{B \vdash A \vee B}\,R_{\vee_2}}{A \vee B \vdash A \vee B}\,L_\vee$$

Once again, this reduction in the complexity of the Reflexivity rule can be understood as establishing that $A \vee B$ follows from itself, given the rules for

disjunction, and thus, that it is sufficiently strong in its function as a premise of good inference, given to what is required for it to function as a conclusion of a good inference. The reduction for the conditional goes as follows:

$$\dfrac{}{A \to B \vdash A \to B}\ \text{Reflex.} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Reflex.} \quad \dfrac{}{B \vdash B}\ \text{Reflex.}}{A, B \vdash B}\ \text{Mo.}}{A \to B, A \vdash B}\ L_{\to}}{A \to B \vdash A \to B}\ R_{\to}$$

Finally, we have the following transformation for negation:

$$\dfrac{}{\neg A \vdash \neg A}\ \text{Reflex.} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Reflex.}}{\neg A, A \vdash}\ L_{\neg}}{\neg A \vdash \neg A}\ R_{\neg}$$

These transformations show that, though we have allowed instances of the Reflexivity axiom, $A \vdash A$, where $A$ is logically complex, we actually could have restricted the axiom schema to permit only cases in which $A$ is *atomic*. Thus, just as Cut is eliminable, *non-atomic Reflexivity* is eliminable from this system. Once again, these reductions can be understood as showing that, given the rules for introducing sentences with these logical connectives as premises or conclusions, their role as a premise is not too weak relative to their role as a conclusion.

Having shown that reductions in Reflexivity complexity are possible for all of our logical connectives, let us now the attempt at such a reduction for the connective **tunk**:

$$\dfrac{}{A\ \mathbf{tunk}\ B \vdash A\ \mathbf{tunk}\ B}\ \text{Reflex.} \quad \rightsquigarrow^{??} \quad \dfrac{\dfrac{\dfrac{}{A \vdash A}\ \text{Reflex.} \quad B \vdash A}{A\ \mathbf{tunk}\ B \vdash A}\ L_{\mathbf{tunk}} \quad \dfrac{\dfrac{A \vdash B \quad \dfrac{}{B \vdash B}}{A\ \mathbf{tunk}\ B \vdash B}\ L_{\mathbf{tun}}}{A\ \mathbf{tunk}\ B \vdash A\ \mathbf{tunk}\ B}\ R_{\mathbf{tu}}$$

Here, we need to use the non-axioms $B \vdash A$ and $A \vdash B$ to derive the axiom, $A\ \mathbf{tunk}\ B \vdash A\ \mathbf{tunk}\ B$. Alternatively, working backwards and applying the left **tunk** rule first, we still arrive at a tree whose leaves contain non-axioms:

$$\frac{}{A \textbf{ tunk } B \vdash A \textbf{ tunk } B}\ \text{Reflex.} \qquad \rightsquigarrow ?? \qquad \frac{\dfrac{\dfrac{}{A \vdash A}\ \text{Reflex.} \quad A \vdash B}{A \vdash A \textbf{ tunk } B}\ R_{\textbf{tunk}} \quad \dfrac{B \vdash A \quad \overline{B}}{B \vdash A \textbf{ tunk}}}{A \textbf{ tunk } B \vdash A \textbf{ tunk } B}$$

Thus, the only way to derive $A$ **tunk** $B \vdash A$ **tunk** $B$, an axiom of this system, is with *non-axioms*. Intuitively, $A$ **tunk** $B$ doesn't follow from itself because it's role as a *premise* is *weaker* than as a conclusion.

The above reductions of Cut complexity and Reflexivity complexity for conjunction, disjunction, the conditional, and negation establish that the rules for these connectives are *harmonious*. That is the role of these connectives as a conclusion and a premise, as determined by the right and left rules of the sequent calculus, fit together harmoniously, with the conclusory role of a sentence containing one of these connectives being neither too strong nor too weak relative to its premisory role. In addition to harmony, we can note that all of these rules are *pure*, *simple*, and *sheer*. Thus, to use the term we used in the previous chapter in connection to natural deduction, all of these sequent rules are *immaculate*.

## 3.5 The Move to Multiple Conclusions

We've now provided a sequent calculus that gives rules for negation, conjunction, disjunction, and the conditional, and we've shown that the rules we've given for these connectives (unlike those for **tonk** and **tonk** fit together harmoniously. This sequent calculus might thus be understood as providing an inferentialist account of the meanings of these connectives. However, there's an issue, and it's one that you might have foreseen given our issue in the previous chapter: the sequent system we've provided does not yield the consequence relation of *classical* logic. Rather, it is a sequent system for *intuitionistic* logic. There is, for instance, no way to prove $\vdash p \vee \neg p$ or $\neg(p \wedge q) \vdash \neg p \vee \neg q$ in this system.

Now, in the case of natural deduction, the move from intuitionistic to classical logic presented a serious technical difficulty, given the requirement of harmony. In the case of the sequent calculus, however, there is a simple and technically elegant modification of the sequent calculus we've just given so

that it yields classical logic: we simply allow sequents to have *multiple conclusions*. Thus, for instance, we generalize the conjunction rules to allow multiple conclusions as follows (where $X$ and $Y$ are both sets of sentences):

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \mathrm{L}_{\wedge} \qquad\qquad \frac{X \vdash A, Y \quad X \vdash A, Y}{X \vdash A \wedge B, Y}\ \mathrm{R}_{\wedge}$$

Crucially, however, the two sides of a multiple conclusion are not treated in the same way. Whereas the *premises* of a multiple conclusion sequent of the form $X \vdash Y$ are collected *conjunctively* (such that, in the context of the this sequent, $X$ is equivalent to the *conjunction* of every sentence in $X$), the *conclusions* of a multiple conclusion sequent of the form $X \vdash Y$ are collected *disjunctively* (such that, in the context of this sequent, $Y$ is equivalent to the *disjunction* of every sentence in $X$). So, $A, B \vdash C, D$ is logically equivalent to $A \wedge B \vdash C \vee D$. We will return to conceptual issues involved in the interpretation of multiple conclusion sequents, so-understood, shortly. First, however, let us see how, technically, modifying the calculus so as to allow for multiple conclusions yields classical logic.

Let us consider first the proof of $\neg(p \wedge q) \vdash \neg p \vee \neg q$, which is classically valid but not intuitionistically valid. Allowing for multiple conclusions, we can prove it like so:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\ \text{Reflex.}}{\vdash p, \neg p}\ R_{\neg}}{\vdash p, \neg p \vee \neg q}\ R_{\vee_2} \qquad \dfrac{\dfrac{\dfrac{}{q \vdash q}\ \text{Reflex.}}{\vdash q, \neg q}\ R_{\neg}}{\vdash q, \neg p \vee \neg q}\ R_{\vee_2}}{\vdash p \wedge q, \neg p \vee \neg p}\ R_{\wedge}}{\neg(p \wedge q) \vdash \neg p \vee \neg q}\ L_{\neg}$$

This proof perhaps feels a bit strange, pushing formulas to the right side of the turnstile only to move one of them back with the left negation rule. For a proof that is even stranger in this multiple conclusion context, consider that $p \vee \neg p$ is a classically tautology, and so a sequent calculus for classical logic must enable us to prove it. Allowing multiple conclusions, we can prove it like so:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{p \vdash p}\,\text{Reflex.}}{\vdash p, \neg p}\,R_{\neg}}{\vdash p \vee \neg p, \neg p}\,R_{\vee_1}}{\vdash p \vee \neg p, p \vee \neg p}\,R_{\vee_2}}{\vdash p \vee \neg p}\,\text{Contr.}$$

Here, we start by using the right negation rule to push $p$ from the left side of the sequent to the right side to obtain $\vdash p, \neg p$. We then use the right disjunction rules *twice*, once on each of the two sentences, to obtain $\vdash p \vee \neg p, p \vee \neg p$ . Finally, we use a structural rule called *Contraction*, which tells us that if we have the same sentence twice on one side of the sequent, we can delete the extra copies. Allowing this rule on both sides of the sequent, we have the following two rules:

$$\dfrac{X \vdash A, A, Y}{X \vdash A, Y}\,R_{\text{Contr.}} \qquad\qquad \dfrac{X, A, A \vdash Y}{X, A \vdash Y}\,L_{\text{Contr.}}$$

Now, said above that the $X$ and $Y$ on both sides of the sequent were *sets* of sentences. Being a bit more precise now, we should actually take them to be *multisets* of sentences. A multiset is just like a set (notably, the order of elements still makes no difference) but, whereas the number of copies of the same thing makes no difference in the context of a set, it does make a difference in the context of a multiset. So, for instance, whereas $\{a, b\}$ and $\{a, a, b\}$ denote the very same *set*, they denote two distinct *multisets*: the latter but not the former contains two copies of $a$. In what follows, I will, for the most part, continue to be a bit loose about the distinction between sets and multisets, treating $X$ and $Y$ as sets, but there will be some places in which we will need to be careful that what we are really working with are actually multisets.

Now, the above proof is a bit strange in requiring us to derive $p \vee \neg q$ twice on the right side of the turnstile. While this is a legitimate proof in this system, if we already have multiple conclusions at our disposal, however, we can tweak the right rules for disjunction, combining them into a single rule that will facilitate much nicer proofs. As we've already said, in a sequent of the form the form $X \vdash Y$, the elements of $X$ are collected conjunctively, but the elements of $Y$ are collected disjunctively. So, a much more elegant right disjunction rule, perfectly analogous to our left conjunction rule, is the following:

$$\frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\ \mathrm{R}_{\vee}$$

With this rule, we can derive $\vdash p \vee \neg p$ quite simply as follows:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\ \text{Reflex.}}{\vdash p, \neg p}\ R_{\neg}}{\vdash p \vee \neg p}\ R_{\vee}$$

Note that this proof, which establishes that $p \vee \neg p$ is a *tautology*, is now perfectly analogous to the proof of $p \wedge \neg p \vdash$, which establishes that $p \wedge \neg p$ is a *contradiction*, which goes as follows:

$$\dfrac{\dfrac{\dfrac{}{p \vdash p}\ \text{Reflex.}}{p, \neg p \vdash}\ L_{\neg}}{p \wedge \neg p \vdash}\ L_{\wedge}$$

With this disjunction rule, which is exactly analogous to the conjunction rules, the duality of conjunction and disjunction, corresponding to two sides of the turnstile, becomes very apparent.

## 3.6 A Multiple Conclusion Sequent Calculus

Let us now put forward a multiple conclusion sequent calculus with this new disjunction rule. We'll start by slightly modifying our axiom schema. You might have noticed in the proofs above that we can do all our applications of Monotonicity at the *beginning* of the proof, immediately after applying the Reflexivity Axiom. Accordingly, it will be a bit technically nicer to *combine* Reflexivity and Monotonicity (on both the left and the right) into a single axiom schema, which we'll call *Containment*.

$$\frac{}{X, A \vdash A, Y}\ \text{Containment}$$

With this axiom schema, we build in all the Monotonicity we need for classical logic right at the outset, so we never need to worry about applying Monotonicity. In fact, in this new sequent calculus, Containment will be the *only* structural rule we need: we don't need Cut (though, to be clear, neither does

the earlier sequent calculus), and we don't even need Contraction. Moreover, we can restrict the Containment axiom to just cases where $A$ is atomic and $X$ and $Y$ contain only atomic sentences.

The operational rules are just the multiple conclusion versions of the rules for the single conclusion sequent calculus, with the modified rule for disjunction introduced in the previous section. So the rules are the following:

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \mathrm{L}_\neg \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \mathrm{R}_\neg$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \mathrm{L}_\wedge \qquad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\ \mathrm{R}_\wedge$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y}\ \mathrm{L}_\vee \qquad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\ \mathrm{R}_\vee$$

$$\frac{X \vdash A, Y \quad X, B \vdash Y}{X, A \to B \vdash Y}\ \mathrm{L}_\to \qquad \frac{X, A \vdash B, Y}{X \vdash A \to B, Y}\ \mathrm{R}_\to$$

I will call this sequent calculus "K," after Oiva Ketonen, who originally proposed it in 1945.

There are a number of very nice features of this new sequent calculus, K. I mentioned above that, in general, the way to use a sequent calculus is *backwards*, starting with the sequent you want to prove and applying the rules in reverse to decompose it until you arrive at axioms. However, in the original calculus, which had multiple right rules for disjunction, we had to actually make some decisions in applying the rules backwards. Consider the simple proof of $p \vee q \vdash p \vee q$ from atomic axioms. Working backwards to derive this sequent from axioms, if we applied either of right disjunction first rules to derive $p \vee q \vdash p$ or $p \vee q \vdash q$, we'd be left with an invalid sequent. After all, $p$ doesn't follow from $p \vee q$. So, we'd have to make sure to first apply the *left* disjunction rule to derive $p \vdash p \vee q$ and $q \vdash p \vee q$ as premises, from which we could apply the first right disjunction rule and left disjunction rule respectively to arrive at the axioms $p \vdash p$ and $q \vdash q$. In contrast to our original sequent calculus, the notable feature of this new sequent calculus is that we never need to make any choices. That is, this sequent calculus, in fact, provides a *decision procedure* for classical propositional logic. That is, if you want to know whether

or not some inference is classically valid, you can just start with the sequent expressing it and, without thinking, decompose it according to the rules of this sequent calculus

The core feature of the sequent calculus that makes it such that root-first proof search always works is that the rules of it are all *invertible*. That is, they all have the following property:

> **Invertibility:** A rule of the form
>
> $$\frac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2 \quad \ldots X_n \vdash Y_n}{X \vdash Y}$$
>
> is *invertible* just in case, whenever the conclusion sequent $X \vdash Y$ is derivable, then each of the premises $X_1 \vdash Y_1, \ldots, X_n \vdash Y_n$ is also derivable. A sequent calculus is invertible if all of its rules are.

To see that the rules of the original sequent calculus are not invertible, consider that $\vdash p \vee \neg p$ is derivable in the multiple conclusion sequent calculus with the original disjunction rule, and the premises from which this formula could be derived (from either of the right disjunction rules) are $\vdash p$ and $\vdash \neg p$. However, the sequent calculus cannot derive either $\vdash p$ or $\vdash \neg p$. The new rules, on the other hand, *are* invertible, since, whenever $\vdash p \vee \neg p$ is derivable, so is $\vdash p, \neg p$.

Invertibility can be proven in different ways. In the standard proof of invertiblity, we show that by an induction on proof height that, if we can derive $X \vdash Y$, we can derive any set of premises from which $X \vdash Y$ can be derived with proofs that are *no longer* than that of the original sequent. This is technically significant, and the proof can be found in various textbooks. However, a simple way to see that Invertibility holds in the new calculus is to note that we could add the Cut rule to this sequent calculus without being able to prove anything we could not already prove (this is a consequence of the Cut elimination theorem, as explained in the section below), and, with Cut, given the conclusion sequent of any rule, we can always derive the premise sequents.[7] Consider first disjunction, with the new right disjunction rule:

[7] This is, in fact, the way that Ketonen first proved it himself.

$$\dfrac{X \vdash A \vee B, Y \qquad \dfrac{\dfrac{}{X, A \vdash A, B, Y}\,\text{Co.} \quad \dfrac{}{X, B \vdash A, B, Y}\,\text{Co.}}{X, A \vee B \vdash A, B, Y}\,L_\vee}{X \vdash A, B, Y}\,\text{Cut}$$

Here we show that, given Containment and Cut, if we have the conclusion sequent of our right disjunction rule, we can derive the premise sequent. Note that, here, we use the *multiple conclusion* version of Cut:

$$\dfrac{X \vdash A, Y \quad X', A \vdash Y'}{X, X' \vdash Y, Y'}\,\text{Cut}$$

Now consider the left disjunction rule:

$$\dfrac{X, A \vee B \vdash Y \quad \dfrac{\dfrac{}{X, A \vdash A, B, Y}\,\text{Co.}}{X, A \vdash A \vee B, Y}\,R_\vee}{X, A \vdash Y}\,\text{Cut} \qquad \dfrac{X, A \vee B \vdash Y \quad \dfrac{\dfrac{}{X, B \vdash A, B, Y}\,\text{Co.}}{X, B \vdash A \vee B, Y}\,R_\vee}{X, B \vdash Y}\,\text{Cut}$$

So, if we have a disjunction on the left, then, given Cut (and Contraction), we can derive both of the premises needed to derive this disjunction.

The proof of invertibility of the conjunction rules are exactly dual to the disjunction rules:

$$\dfrac{X \vdash A \wedge B, Y \quad \dfrac{\dfrac{}{X, A, B \vdash A, Y}\,\text{Co.}}{X, A \wedge B \vdash A, Y}\,L_\wedge}{X \vdash A, Y}\,\text{Cut} \qquad \dfrac{X \vdash A \wedge B, Y \quad \dfrac{\dfrac{}{X, A, B \vdash B, Y}\,\text{Co.}}{X, A \wedge B \vdash B, Y}\,L_\wedge}{X \vdash B, Y}\,\text{Cut}$$

$$\dfrac{X, A \wedge B \vdash Y \qquad \dfrac{\dfrac{}{X, A, B \vdash A, Y}\,\text{Co.} \quad \dfrac{}{X, A, B \vdash B, Y}\,\text{Co.}}{X, A, B \vdash A \wedge B, Y}\,R_\wedge}{X, A, B \vdash Y}\,\text{Cut}$$

The proof of the invertibility of the conditional rules is similar and left as an exercise. Finally, the invertibility of the negation rules is shown as follows:

$$\dfrac{X, \neg A \vdash Y \quad \dfrac{\dfrac{}{X, A \vdash A, Y}\,\text{Co.}}{X \vdash A, \neg A, Y}\,R_\neg}{X \vdash A, Y}\,\text{Cut} \qquad \dfrac{X \vdash \neg A, Y \quad \dfrac{\dfrac{}{X, A \vdash A, Y}\,\text{Co.}}{X, A, \neg A \vdash Y}\,L_\neg}{X, A \vdash Y}\,\text{Cut}$$

Thus, all of the rules of this new sequent calculus, disjunction included, are invertible.

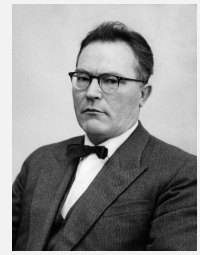
Invertibility is not just a technically significant feature of this sequent calculus. It is also philosophically notable in the context of proof theoretic semantics, for it enables us to think of the top sequents of the rules as providing both *necessary and sufficient* conditions for when a logically complex formula figures as a premise or a conclusion of a good inference. In this way, we can think of the left and right rules of the sequent calculus as providing proof-theoretic *definitions* of the connectives they introduce. Consider again the conjunction rules:

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \mathrm{L}_{\wedge} \qquad\qquad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\ \mathrm{R}_{\wedge}$$

Given the invertibility of these rules, we can say that $A \wedge B$ functions as a premise of a good inference, relative to a pair of premises/conclusions $(X, Y)$, *just in case* $A$ along with $B$ functions as premises of a good inference relative to $(X, Y)$. Likewise, we can say that $A \wedge B$ functions as a *conclusion* of a good inference, relative to a pair of premises and conclusions $(X, Y)$, *just in case* $A$ functions as a conclusion of a good inference relative to $(X, Y)$ and $B$ functions as a conclusion of a good inference relative to $(X, Y)$. Similarly, for disjunction, the conditional, and negation.

***Historical Note:***

This wonderfully elegant sequent calculus for classical logic was developed by Oiva Ketonen. In 1938 Ketonen, a doctoral student at the University of Helsinki in Finland, went to Gottingen to study with Gerhard Gentzen. In his 1944 dissertation, he showed how, by tweaking the rules of Gentzen's classical sequent calculus, one could obtain a sequent calculus in which all of the rules are invertible.



Recently, Ketonen's work has been brought to the attention of philosophers and logicians by Sara Negri and Jan Von Plato, who were actually Ketonen's colleques

at the University of Helsiki. They have done substantial work in proof theory developing and extending the sort of proof-systems proposed by Ketnonen (as well as translating Ketonen's original work into English). For a more technical exposition of the features of this sequent calculus, and full proofs of these features, I recommend Negri and von Plato's book, *Structural Proof Theory*.

## 3.7 Eliminability, Admissibility, and Derivability

We referred earlier to the proof that the rule of Cut is *eliminable* from the sequent calculus containing it. We also showed that the rule of Reflexivity can be restricted to atomics. The way we originally formulated the sequent calculus was, to first, stipulate that it contains these rules, and then state that, in fact, it doesn't need them. That is, we showed that these rules have the following property:

> **Eliminability:** A rule $\mathcal{R}$ belonging to a sequent calculus $\mathcal{C}$ of the form
>
> $$\frac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2 \quad \ldots X_n \vdash Y_n}{X \vdash Y}$$
>
> is *eliminable* just in case, in the sequent calculus $\mathcal{C}_-$ resulting from subtracting $R$ from $\mathcal{C}$, if $X_1 \vdash Y_1, X_2 \vdash Y_2, \ldots X_n \vdash Y_n$ are provable, then so is $X \vdash Y$.

Alternatively, however, we could have followed the approach we took to our second sequent calculus and stated the sequent calculus *without* the Cut rule and showed that adding Cut would not enable us to prove anything we could not already prove. That is, we could have shown that, in the calculus without the Cut rule, the Cut rule is nevertheless *admissible:*

> **Admissibility:** A rule of the form
>
> $$\frac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2 \quad \ldots X_n \vdash Y_n}{X \vdash Y}$$

> is *admissible* in a sequent calculus $C$ just in case, if $X_1 \vdash Y_1, X_2 \vdash Y_2, \ldots X_n \vdash Y_n$ are provable in $C$, then so is $X \vdash Y$.

Thus, admissibility of a rule in a sequent calculus means that you can add the rule to the sequent calculus and it won't enable you to prove anything that you couldn't already prove. Admissiblity and eliminability are obviously very closely related. In general, $\mathcal{R}$ is admissible in $C$ just in case $\mathcal{R}$ is eliminable in $C + \mathcal{R}$. To give just one more example of admissibility, we above pointed out that the rules of this new multiple conclusion sequent calculus were *invertible*. What that means, technically, is that for any of the operational the rules of the sequent calculus, its converse (i.e. the bottom-to-top direction of the rule) is *admissible*. That is, even though the converse of these rules are not official rules of the sequent calculus, we could add them to the sequent calculus and doing so won't enable us to prove anything we couldn't already prove.

A related, but crucially distinct notion is *derivability*. This is the following notion:

> **Derivability:** A rule of the form
>
> $$\frac{X_1 \vdash Y_1 \quad X_2 \vdash Y_2 \quad \ldots X_n \vdash Y_n}{X \vdash Y}$$
>
> is *derivable* in a sequent calculus $C$ just in case $X \vdash Y$ is provable from $X_1 \vdash Y_1, X_2 \vdash Y_2, \ldots X_n \vdash Y_n$ in $C$.

If a rule is derivable in a sequent calculus, you can add it to the sequent calculus and treat applications of that rule as shorthand for the full proof of the conclusion from the premises. Now, if a rule is *derivable* in a sequent calculus, then it is certainly *admissible* in that sequent calculus. However, the converse doesn't necessarily hold. That is, there might be some rules that are admissible in some sequent calculus but not derivable in that sequent calculus. To see how this might be so, consider again our new multiple conclusion sequent calculus which contains the following rule for disjucntion:

$$\frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\ R_\vee$$

Now consider again the pair of disjunction rules of the original sequent calculus:

$$\frac{X \vdash A, Y}{X \vdash A \vee B, Y}\, R_{\vee_1} \qquad\qquad \frac{X \vdash B, Y}{X \vdash A \vee B, Y}\, R_{\vee_2}$$

If the sequent calculus contains the structural rule of *Weakening*, then the each of the second rules are obviously derivable from the first rule as follows:[8]

$$\dfrac{\dfrac{X \vdash A, Y}{X \vdash A, B, Y}\,\text{Weakening}}{X \vdash A \vee B, Y}\, R_{\vee} \qquad\qquad \dfrac{\dfrac{X \vdash B, Y}{X \vdash A, B, Y}\,\text{Weakening}}{X \vdash A \vee B, Y}\, R_{\vee}$$

However, if the sequent calculus does *not* contain Weakening, then there is no way to derive either the second rules with the use of the first rule. Nevertheless, one feature of the sequent calculus K is that, even if it does not contain Weakening, Weakening is nevertheless *admissible*. It follows that the original disjunction rules are admissible in K, but not derivable. We will return to the significance of this point when we start considering the integration of non-logical axioms into a sequent calculus in Chapter 6.

## 3.8 The Issue with Interpretability

Unlike the natural deduction system for classical logic laid out in the previous chapter, there is no technical shortcoming of this sequent calculus for classical logic. To use the phrase we coined in the previous chapter, the rules are all proof-theoretically immaculate. Moreover, the invertible sequent calculus we introduced above can be counted as among the most technically elegant proof systems for classical logic. The main issue with this system, however, is not technical, but *conceptual*. To explicate the conceptual issue here, we need to zoom out and consider once again the project of logical inferentialism in the context of which this system might be proposed.

[8] What about the other direction? As the following proof shows, the new rule is derivable from the old two rules just in case the sequent calculus contains the structural rule of Contraction:

$$\dfrac{\dfrac{\dfrac{X \vdash A, B, Y}{X \vdash A \vee B, B, Y}}{X \vdash A \vee B, A \vee B, Y}}{X \vdash A \vee B, Y}\,\text{Contr.}$$

Note that if the original sequent calculus *doesn't* contain Contraction, then Contraction is not admissible either, since, as we've seen, it is needed, for instance, to prove $\vdash p \vee \neg p$.

An inferentialist account of the meanings of the connectives aims to articulate their meaning in terms of the rules governing their use in *inferences*. At this point, we should be more explicit about just what conception of "inferential rule" we're deploying here. First off, insofar as these are understood as rules governing the *use* of sentences, they should really be understood as rules for *asserting* sentences. After all, when one actually uses (i.e. utters) a standalone declarative sentence, assertion is generally the speech act one performs in doing so. Now, following Brandom [2], we might think that there are two fundamental and different ways in which we might think about an inferential rule that might be expressed with a sequent of the form $A \vdash B$: a *permissive* way and a *committive* way. On a *permissive* conception, an inferential rule of the form $A \vdash B$ says that if you licitly assert $A$ (i.e. assert $A$ with entitlement to so), you're *entitled* to assert $B$. On a *committive* conception, $A \vdash B$ says that if you assert $A$, you're *committed* to asserting $B$. Of course, being committed to asserting some sentence doesn't mean that one must actually *assert* that sentence, unprompted (if it did, we'd be stuck asserting sentences all day), but it does mean that one can be *held* to asserting it if appropriately prompted to do so. Thus, if $B$ is a committive consequence of $A$, and you assert $A$ but you *refuse* to assert $B$ when appropriately prompted to do so, you're, as Ripley [8] puts it, normatively "out of bounds." One might opt to think of $A \vdash B$ in terms of either of these notions of consequence. Note now that both conceptions extend naturally to sequents with multiple *premises*. For instance, to take just the committive notion, we can read $A, B, C \vdash D$ as saying that asserting $A$, asserting $B$, and asserting $C$ commits one to asserting $D$. Notably, asserting $A$ and asserting $B$ is distinct from asserting $A \wedge B$, though logically equivalent to it, and this logical equivalence is precisely what is formally captured in a sequent calculus. The same, however, cannot be said about sequents with multiple *conclusions*.

Consider the sort of sequent we can derive in the multiple-conclusion sequent calculus we've laid out:

$$A \vee B \vdash A, B$$

Note first that this sequent can't be read as saying that, if one asserts to $A \vee B$ then one is either committed to asserting $A$ or committed to asserting $B$, since one

could assert $A \vee B$, yet be committed neither to asserting $A$ nor to asserting $B$. I might, for instance, assert "There's an odd number or an even number of blades of grass in Central Park," but I'm neither committed to "There's an odd number of blades of grass in central park" nor "There's an even number of blades of grass in central park." Note also that, insofar as we want sequents like $A \vee B \vdash A, B$ to non-circularly account the meaning of disjunction in inferential terms, we can't analyze the multiple conclusions of this sequent in terms of commitment to a disjunction. Once again, we have no problem reading $A, B \vdash A \wedge B$ as saying that asserting $A$ along with asserting $B$ commits one to asserting $A \wedge B$, and we might take this fact to in part account for the meaning of conjunction in inferential terms. Yet there is no straightforward reading of $A \vee B \vdash A, B$ as expressing a relation of committive consequence between the assertion of $A \vee B$, the assertion of $A$, and the assertion of $B$.

Now, one proposal for understanding multiple conclusions is the method of *proof by cases*, often exhibited in a mathematical proof.[9] Often, in the context of a mathematical proof, one will, given what one has already established, consider the range of possible cases. Thus, for instance, one might reason something like the following: given $A$, $B$, and $C$, there are three cases to consider, $D$, $E$, and $F$. We might then go to reason, for instance, that $G$ follows from $D$, $G$ follows from $E$, and $G$ follows from $F$, and thus, $G$ follows from $A$, $B$, and $C$. This seems to be an informal statement of a kind of implication in which the premises are collected conjunctively and the conclusions are collected disjunctively. Thus, $A \vee B \vdash A, B$ might be read as saying, given $A \vee B$ as a premise, there are two cases to consider: $A$ and $B$. In this way, $A$ and $B$ could be understood as conclusions "following from" $A \vee B$ as possible cases to consider, and, this is plausibly at least *a* sense of "following from" in which we can say that the conclusions of a multiple conclusion sequent "follow from" the premises.

Still, while this interpretation is intelligible, it is still takes us some ways from our natural understanding of consequence or the core theoretical notions of consequence such as permissive or committive consequence that figure in the sort of normative inferentialist theory developed by such authors as Brandom. Moreover, it's not clear that this interpretation will really give us a non-circular

[9] This is suggested by Smiley and. Thanks to Michael Kremer for pushing me on this point

account of, for instance, the meaning of disjunction. Do we *really* have a grip on what it is for $A$ and $B$ to constitute the "cases" in the context of the sequent $A \vee B \vdash A, B$ independently of our grip on disjunction? It is at least not obvious that we do. While these are far from decisive reasons to discount the "proof by cases" reading of multiple conclusion sequents in the context of an inferentialist theory of meaning, they do at least provide us with some reason to look for alternatives. In the chapter following next, we will consider a *bilateral* interpretation of multiple conclusion sequents where $X \vdash Y$ is read as saying that asserting everything in $X$ and denying everything in $Y$ is incoherent or "out of bounds." First, however, in the next chapter, we will return to consider natural deduction and look specifically at *bilateral* natural deduction systems where we have rules governing the assertions and denials of sentences.

## 3.9 Exercises

1. Provide

2. Provide reductions of Cut complexity and Reflexivity complexity for the new multiple conclusion disjunction rules, introduced in SEction, to show that these rules are harmonious.

3. Use the method shown in Section to show that the rules for the conditional are invertible .