Bringing Bilateralisms Together:
A Unified Framework for Inferentialists

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0 Introduction

Inferentialism aims to account for the meanings of linguistic expressions in terms of the inferential rules governing their use. One of the main formal developments in the inferentialist program over the past few decades has been bilateralism in proof-theoretic semantics, according to which assertion and denial are taken to be equally basic in providing an account of the meanings of expressions in terms of the rules governing their use in proof systems. Bilateralists are split, however, as to what this actually entails. Some bilateralists, following Greg Restall (2005), use bilateralism to interpret existing proof systems, such as Gentzen's multiple conclusion classical sequent calculus. Other bilateralists, following Ian Rumfitt (2000), develop distinctively bilateral proof systems in which formulas are positively or negatively signed. In this paper, I explore the respective virtues of these two styles of bilateralism in the context of the broader inferentialist program. I argue that, considered in this application, both forms of bilateralism have distinctive virtues. On the one hand, the single conclusion systems developed by Rumfitt-style bilateralists enable a straightforward specification of the discursive significance of a sentence in terms of the inferential conditions and consequences of its use. On the other hand, the multiple conclusion sequent systems readily made sense of by Restall's bilateralism are particularly well-suited to accommodating defeasible material inferential relations which are an essential component of an inferentialist theory. After laying out these respective virtues, I show how these forms of bilateralism can be brought together in a single bilateral framework that has the virtues of both.

While this formal bridge between bilateralisms is itself of significant interest to bilateralists of different stripes, the main upshot of this paper is a new kind of bilateral framework that is uniquely suited for inferentialist semantics.

1 Two Ways to Be a Bilateralist

A bilateral conception of a logic takes as basic two opposite ways of being related to the propositions expressed by the sentences with which the logic is concerned. On the standard way of thinking about bilateralism, these "two ways" are two opposite speech acts: assertion and denial.[1] These two acts might be understood as two opposite *stances* that one might take towards a given proposition in discourse: a positive stance and a negative stance. However one opts to philosophically spell out these two opposite stances, the crucial logical thought is that taking the negative stance towards some proposition is conceived as *distinct from* though *logically equivalent to* taking the positive stance towards its negation. That is, denying A is distinct from but logically equivalent to asserting $\neg A$. With the position construed, prominent bilateralists include Greg Restall, David Ripley, Ian Rumfitt, Nissim Francez, and Luca Incurvati and Julian Schlöder. Dividing these bilateralists, however, are two very different ways to be a bilateralist, two distinct styles of bilateralism.

The first style of bilateralism, put forward by Restall (2005, 2009, 2013) and influentially developed by Ripley (2014, 2015, 2017), takes existing proof systems, such as Gentzen's sequent calculus for classical logic, and interprets them in a bilateralist fashion.[2] What is notable about Gentzen's classical sequent calculus is that the sequents one manipulates through the use of the calculus have *multiple conclusions*. The standard way of thinking about a multiple conclusion sequent of the form $X \vdash Y$, where X and Y are both sets of

[1]There is some variation among authors in exactly how these two ways are specified. Rumfitt (2000) speaks of "affirmation" and "denial," understood as the opposite acts of saying "Yes" to the question of whether the proposition is true or saying "No" to it. Others, such as Smiley (1996), speak of "rejection" as the negative stance, rather than denial. Little hangs on these terminological variations for our purposes here.

[2]For other notable proponents and developers of this style of bilateralism, see Tanter (2021a, 2021b), Rosenblatt (2019), Hlobil (2019, 2023), Hlobil and Brandom (2024), Francez (2019), and Hjortland (2014).

sentences, is that the elements of X, collected conjunctively, *imply* the elements of Y, collected disjunctively.[3] Critics of multiple conclusion sequent calculi have argued that "arguments" with multiple conclusions, understood in this way, are too far from our ordinary practices of reasoning to be appealed to in providing a proof-theoretic account of the meanings of the logical connectives.[4] In response to these concerns, Restall proposes a reading of multiple conclusion sequents according to which the turnstile plays the role not of separating *premises* from *conclusions* but of separating *assertions* from *denials*. On this reading, a sequent of the form $X \vdash Y$ says that the position consisting in asserting everything in X and denying everything in Y is incoherent or "out of bounds." This interpretation enables us to explicate and justify the standard structural and operational rules of classical logic in an intuitive way. For instance, we can understand the axiom schema of Identity, $X, A \vdash A, Y$, as telling us that it's always out of bounds to assert and deny a single sentence A, no matter what else we assert or deny. For operational rules, consider the negation rules of the classical sequent calculus:

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \mathrm{L}_\neg \qquad\qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \mathrm{R}_\neg$$

On the bilateralist interpretation, the left rule says that if the position consisting in asserting everything in X, denying everything in Y, and denying A is out of bounds, then the position consisting in asserting everything in X, denying everything in Y, and asserting $\neg A$ is out of bounds. The right rule says, similarly, that if, in the context of some position, asserting A is out of bounds, then, in the context of that position, denying $\neg A$ is out of bounds. So, understanding the significance of speech acts in terms of their contribution to the (in)coherence of positions, these rules tells us that an assertion of $\neg A$ has the same significance as a denial of A, and a denial of $\neg A$ has the same significance as an assertion of A. In the same way, all of the other rules of a multiple conclusion sequent system like Gentzen's classical sequent calculus, LK, can be intuitively explained. In

[3]See Shoesmith and Smiley (1978) for the classical articulation of this standard interpretation.

[4]See, for instance, Rumfitt (2000, 2008) and Steinberger (2011) for criticisms of this sort.

this way, Restall's bilateralism presents itself as a way of vindicating classical logic.

The second style of bilateralism, put forward Rumfitt (2000) and influentially developed by Francez (2014, 2015) and Incurvati and Schlöder (2017, 2019, 2023), involves constructing proof systems in a distinctively bilateralist manner, providing rules for manipulating positively or negatively signed formulas.[5] Dummett (1991) famously argues that, if we want to think of the meanings of the logical connectives in terms of the rules governing their use in proofs in natural deduction, we should be intuitionists rather than classicalists, since it is intuitionistic natural deduction rather than classical natural deduction that displays that proof-theoretic virtue of *harmony*, with the introduction and elimination rules fitting together as they ought, with each set of rules being neither too strong nor too weak relative to the other.[6] In response to Dummett, Rumfit (2000), drawing on prior work from Smiley (1996), shows that if one has a natural deduction system that contains not just rules for *asserting* sentences but rules for *denying* sentences as well, then it is easy to arrive at a harmonious natural deduction system for classical logic. In such a system, a well-formed formula must be prefaced with a positive or negative force-marker, expressing either assertion or denial. Thus, the assertion of a sentence A might be written as $+\langle A \rangle$, and the denial of A can be written as $-\langle A \rangle$.[7] Rather than interpreting existing proof systems with the notions of assertion and denial, we construct proof systems in which signs expressing assertion and denial directly figure.

[5]Notably, Incurvati and Schlöder (2017, 2018, 2023), largely in response to Dickie (2010), extend signed bilateralism to *multi*-lateralism, introducing new signs to express "weak" assertion and rejection, but their systems are still "bilateral" in the sense of the term used here in that they are *at least* bilateral. For other notable proponents or developers of this style of bilateralism, see Humberstone (2000), Kürbis (2016, 2019, 2022), Wansing (2016), and Ayhan (2020). There are variations among these bilateralists worth noting. In particular, Wansing (2016), Kürbis (2019), and Ayhan (2020) endorse a somewhat different conception of bilateralism in the development of intuitionistic logic, taking the turnstile itself to be positively or negatively signed, expressing *verification* or *refutation*. Though proponents of this kind of bilateralism take there to be an important conceptual difference between it and standard Rumfitt-style bilateralism (see especially Kürbis (2023) on this point), the signed notation is formally inter-translatable, and so I still take them to fall within this general style.

[6]I leave the notion of harmony informal in this preliminary presentation, as there are many differing conceptions of what, exactly, it amounts to. For discussion, see Steinberger (2011b).

[7]I use angle brackets to be clear that the positive and negative signs always take wide scope over the whole formula. I find it also helps with readability.

For instance, in the natural deduction system proposed by Rumfitt, we have the following rules for negation:

$$\frac{-\langle A\rangle}{+\langle\neg A\rangle}\ {+\neg_I} \qquad \frac{+\langle A\rangle}{-\langle\neg A\rangle}\ {-\neg_I}$$

$$\frac{+\langle\neg A\rangle}{-\langle A\rangle}\ {+\neg_E} \qquad \frac{-\langle\neg A\rangle}{+\langle A\rangle}\ {-\neg_E}$$

These rules likewise formally codify the equivalence of denying A and asserting $\neg A$ as well as the equivalence of asserting A and denying $\neg A$. The sort of reductio that usually figures as the negation introduction rule in Gentzen's natural deduction system, now figures as a distinctively bilateral *structural* rule, called "Smileian Reductio." Where Γ is a set of signed formulas, φ and ψ are signed formulas, and starring a formula yields the oppositely signed formula, the principle is the following:

$$\frac{\Gamma, \varphi \vdash \psi \quad \Gamma, \varphi \vdash \psi^*}{\Gamma \vdash \varphi^*}\ \text{Smileian Reductio}$$

This is a structural rule, analogous to Gentzen's other structural rules in that it makes no reference to any specific connectives, but it is distinctively bilateral in concerning positively and negatively signed formulas. With these negation rules, this added structural rule, and negative conjunction and disjunction rules that exploit the duality of conjunction and disjunction, this style of bilateralism yields a harmonious natural deduction system for classical logic.

So, both bilateralist programs aim, in their own way, to vindicate proof systems for classical logic proposed by Gentzen, the first using bilateralism to interpret Gentzen's multiple conclusion sequent calculus as it stands and the second using bilateralism to tweak Gentzen's natural deduction system to yield a system with desirable formal properties. Despite this striking similarity, there is quite a striking difference between the two programs: they each end up with very different answers to what logic is fundamentally *about*. For the Restall-style bilateralist, logic is fundamentally about *incoherence*, whereas, for the Rumfitt-style bilateralist, logic is fundamentally about *consequence*.[8] This opposition between incoherence and consequence can be made explicit through the use of normative vocabulary developed by Brandom (1994, 2008). The turnstile, on Restall's interpretation of it, can be understood as expressing the negative normative force of *preclusion of entitlement*: a sequent of the form $X \vdash Y$ says that asserting everything in X *precludes one from being entitled* to deny everything in Y. That is what it is for asserting everything in X to be incompatible with denying everything in Y.[9] For a Rumfitt-style bilateralist, on the other hand, the turnstile expresses what one might think a turnstile ought to express: a relation of *consequence*. Once again, Brandom's vocabulary is helpful here. A signed sequent of the form $\Gamma \vdash \varphi$ says that making the moves in Γ (be they assertions or denials) *commits one* to making the move φ.[10]

It's this notion of committive consequence, I take it, that Rumfitt (2008) is speaking of when he speaks of "the force of consequence" which he criticizes Restall's understanding of the turnstile for lacking. In illustrating this notion of force, he considers a hypothetical (or perhaps actual) exchange with a student in one of his seminars:

> What do you mean, you refuse to accept B? You continue to adhere to A, and I've shown you that B follows from A, (80).

Rumfitt takes it that, given the student's acceptance of A and acknowledgment

[8] This characterization of Restall-style bilateralism as a position in which logic is *not* about consequence may be met with some protestation from the Restall-style bilateralist, for they will likely want to say that B's being a consequence of A *is* a matter of the incoherence of asserting A and denying B. However, to say that is to treat "consequence" as a technical term, defined in terms of the notion of incoherence. Thus, the point I am making may be more precisely put by saying that they differ in the primitive conceptual notions deployed to explicate the logical vocabulary: the notion of incoherence or the notion of consequence. It's also worth noting that Rosenblatt (2019) puts forward an anti-sequent calculus to extend Restall-style include coherence and incoherence, but the main point of contrast still holds even distinguishing (in)coherence from consequence.

[9] Ripley (2017) explicitly likens the notion of incoherence codified by multiple conclusion sequents, on Restall's bilateral interpretation of them, to Brandom's notion of "material incompatibility," which Brandom cashes out in terms of this notion of preclusion of entitlement. For Brandom (1994, 2008), a sentence A is incompatible with a sentence B just in case assertion of (or commitment to) *A precludes entitlement* to B.

[10] Incurvati and Schlöder (2017, 2019, 2023) explicitly articulate the rules of their bilateral system as preserving commitment, often with reference to Brandom.

of the fact that *B* follows from *A*, she's *obliged* to accept *B*. That is to say, given the stances that she has taken, she's *committed* to taking this positive stance towards *B*. On Restall's understanding of validity, all one can say here is that she is precluded from being entitled to take the negative stance towards *B*. Of course, Rumfitt acknowledges that is indeed the case, but he thinks it's crucial that we be able to say something stronger here as well. Now, Restall (2013), in defense of his coherence-based account of the logical validity, strikes back against this sort of account in which the turnstile expresses a notion of committive consequence. He says,

> To take an argument to be valid does not mean that when one asserts the premises one should also assert the conclusion (that way lies madness, or at least, making too many assertions). No, to take an argument to be valid involves (at least as a part) the commitment to take the assertion of the premises to stand against the denial of the conclusion, (82).

Clearly, however, this is a jab at a straw man. The notion of being *committed* to asserting some sentence is indeed a kind of obligation, concerning what one *should* do, rather than what one is *precluded* from doing. Crucially, however, it's a sort of *dispositional* obligation, one which can be *triggered* in various circumstances, rather than a *standing* obligation. Specifically, one is committed to asserting the conclusion of a valid argument whose premises one accepts in the sense that one is obligated to assert it *if one is appropriately prompted to do so*, as Rumfitt prompts his student to accept *B* in the above quote.

Now, Restall and Rumfitt each have further arguments for their respective conceptions of logical validity and against each other's.[11] It's not clear, however, that this dispute between the two conceptions of logical validity can be settled in a logical vacuum. That is, if one considers the logical systems apart from any concrete application, it is hard to see any decisive reasons for favoring one conception or the other. However, one of the main broader projects motivating the development of bilateralism of both forms is its potential ap-

[11] For instance, for Restall against Rumfitt, see Restall (2020, 12-14) and Kürbis (2023) for a development of this point, and, for Rumfitt against Restall, see Rumfitt (2015, 51) and Steinberger (2011, 349-353) for a development of this point.

plication in an inferentialist semantics, not just for logical vocabulary, but for natural language in general.[12] When we try to do this, it becomes clear that single conclusion sequents of the sort that figure in Rumfitt-style bilateral systems must play the principal role in formally modeling meaning, from an inferentialist perspective, for only such sequents can be understood as articulating how certain expressions are *to be used*, specifying the inferential conditions and consequences of assertions and denials. However, when it comes to suitability of the actual logical systems to the inferentialist project, the sort of sequent calculi that are readily made sense of by Restall-style bilateralism fare much better, since these sequent systems an straightforwardly accommodate defeasible inferential relations which are essential to an adequate inferentialist theory for natural language. Explaining these respective virtues is the task to which I'll now turn.

2 Respective Virtues for Inferentialist Semantics

Let me start by articulating the reason why many inferentialists have preferred the sort of bilateral systems developed by Rumfitt in the context of articulating an inferentialist theory of meaning. An inferentialist semantics, as the name suggests, aims to articulate the meaning of a linguistic expression in terms of the inferential rules governing its use. In this sense, it is a *use theory of meaning*. Thus, for instance, Brandom begins the précis of *Making It Explicit*, the most philosophically well-developed articulation of an inferentialist theory of meaning, with the sentence, "The book is an attempt to explain the *meanings* of linguistic expressions in terms of their *use*," (1997, 153). Unlike the *descriptive* sort of use theory proposed by Horwich (2004), the sort of use theory proposed by Brandom is a *normative* use theory. As Brandom puts it:

> [The theory] makes essential use of *normative* vocabulary. The practices that confer propositional and other sorts of conceptual content implicitly contain norms concerning how it is *correct* to use expressions, under what circumstances it is *appropriate* to perform various

[12] For expression of this motivation among bilateralists and developments in pursuit of it, see Ripley (2017), Tanter (2021), Francez (2015), and Incurvati and Schlöder (2017, 2019).

> speech acts, and what the appropriate consequences of such performances are, (153).

That is, the sort of inferentialist theory put forward by Brandom articulates the meaning of a sentence in terms of the conditions under which it is *to be used* and the consequences of that use, understood in terms of which other sentences are to be used. The core bit of normative vocabulary that Brandom deploys to articulate a sentence's being such that it is to be used is that of *commitment:* certain uses of certain sentences might *commit* one to uses of other sentences.[13]

The above points have crucial import when it comes to actually spelling out an inferentialist conception of semantic significance. In the context of the Brandomian framework, relations of committive consequence are understood, in the first instance, as principles for keeping discursive score. Such principles, held by the various discursive participants, determine what the utterance of a sentence *does*, normatively speaking, in a discursive practice in which it might be uttered: how the utterance of it changes the "social deontic score," the normative statuses that have been assigned to the various participants of the discursive practice. In Brandom's words:

> The significance of an assertion of p can be thought of as a mapping that associates with one social deontic score—characterizing the stage before that speech act is performed, according to some scorekeeper—the set of scores for the conversational stage that results from the assertion, according to the same scorekeeper (1994, 190).

This conception of the function of inferential norms in determining discursive significance meshes naturally with the interpretation of Rumfitt-style single conclusion bilateral sequents as expressing principles of committive consequence. In general, such sequents tell us how to attribute commitments to someone when they have made some set of moves, updating the social deontic score. Concretely, if $\Gamma \vdash \varphi$ is a derivable sequent, then, if we score someone

[13] Another normative notion that Brandom deploys is that of *entitlement*. There is reason to think that commitment is the more fundamental notion, but, regardless, the the same considerations here apply even in the context of talking about so-called "permissive consequence."

as having made the moves in Γ, we are to score them as committed to φ.[14] In this way, single conclusion bilateral sequents enable us to specify the update in discursive score that ought to take place upon someone's asserting or denying some sentence.[15] Multiple conclusion sequents, interpreted in Restall-style fashion, do no such thing.

To spell out the above point with a concrete example, consider the following sequent, derivable in any classical Rumfitt-style system:

$$+\langle A \vee B\rangle, -\langle A\rangle \vdash +\langle B\rangle$$

Understanding the turnstile here as expressing committive consequence, this says that asserting $A \vee B$ and denying A commits one to asserting B. In this way, a Rumfitt-sytle system, through which we can derive such sequents, tells us how to attribute commitments to someone who asserts a disjunction, given the other things that they've might have asserted or denied. By contrast, consider the sort of sequent we derive in a multiple-conclusion sequent calculus:

$$A \vee B \vdash A, B$$

Note first that this sequent can't be read as saying that, if one is committed to $A \vee B$ then one is either committed to A or committed to B, since one could be committed to neither A nor B yet still be committed to the disjunction $A \vee B$. For instance, I'm committed to "There's an odd number or an even number of blades of grass in Central Park," but I'm neither committed to "There's an

[14] One might wonder whether it a sequent expressing committive consequence should be interpreted in this way or interpreted as saying that someone who is *committed* to the moves in Γ is committed to A. That is indeed how Incurvati and Schlöder (2017, 2019, 2023) think of their proposed bilateral consequence relation, thinking of committive consequence in terms of "commitment preservation," following Brandom (1994). Insofar as we want to accommodate failures of Transitivity considered below, however, we should think that actually *making* a move to which one was antecedently only *committed* might result in one's taking on *new implicit commitments*. So, though I will not develop this conceptual point here, I take there to be reason to not (in all cases) think of committive consequence in terms of commitment preservation.

[15] This basic thought can be made formally precise by using such sequents to define updates in the style of a dynamic semantics of the sort proposed by Veltman (1996). For reasons of space and to avoid distraction from the main philosophical point, I will not introduce such a formal system here.

odd number of blades of grass in central park" nor "There's an even number of blades of grass in central park."[16] This sort of consideration motivates Restall's bilateral reading of multiple conclusion sequents. On Restall's interpretation, the above sequent says that asserting $A \vee B$, denying A, and denying B is out of bounds. That's true, of course, but it doesn't tell us how to attribute commitments to someone who has asserted a disjunction. In general, a multiple conclusion sequent of the form $X \vdash Y$ simply doesn't tell you what to do when someone makes all of the assertions in X. As Restall interprets such a sequent, it tells you that you *can't* score them as *entitled* to all of the denials in Y, but that doesn't amount to telling you what you *should* score them as *committed* to. Such sequents can function to *constrain* scorekeeping practices, but they can't function to *dictate* scorekeeping practices.

I have made the point here appealing to in particular Brandom's conception of discursive significance, the general point is that a Restall-style bilateralism simply does not specify the conditions under which sentences are to be used—it only says which combinations of uses are precluded. Of course, this is not to say that the sort of incoherences that a Restall-style bilateralism concerns itself with are not important. Plausibly, *both* notions ought to be in play in an inferentialist theory. Still, the general thought here is that, on an inferentialist approach to meaning, properly *inferential* relations—relations of *consequence*, properly so-called—ought to play some fundamental role. One might think that these conceptual considerations are decisive in counting against Restall-style bilateralism in the context of an inferentialist account of meaning, and, indeed, many inferentialist such as Steinberger (2011) have thought just that. Cutting in the other direction, however, are certain technical considerations that arise when it comes to actually spelling out an inferentialist theory for natural language beyond the domain of logical vocabulary.[17] In particular, when it comes to accommodating the sorts of *defeasible* inferential relations that

[16]Note also that, insofar as we want sequents like $A \vee B \vdash A, B$ to non-circularly specify the meaning of disjunction in inferential terms, we can't analyze the multiple conclusions of this sequent in terms of commitment to a disjunction.

[17]I mean "logical" broadly here to include such expressions as indicative conditionals, epistemic modals, and probability operators. Thus, for instance, while Incurvati and Schlöder (2023) extend inferentialism to a wide range of broadly logical expressions, they do not extend it to content words and so do not address the difficulties outlined below.

are largely constitutive of the meanings of content words, the sorts of sequent systems made sense of by Restall-style bilateralism seem much better-suited for the job. Let me explain.

Any general inferentialist semantics for natural language must provide an inferential articulation of the meaning of content words such as "red," "colored," "dog," "bird," and so on in terms of the inferential rules governing their use. When inferentialists gesture at the sort of "material inferences" (Sellars 1953) that would figure into an inferentialist theory of content, they typically appeal to strict entailments such as that from "This is scarlet" to "This is red" and that from "This is red" to "This is colored." What inferential relations, however, are constitutive of the meaning of "bird"? Clearly, there are *some* strict entailments, like the one from "Bella's a bird" to "Bella's an animal." However, if one restricts one's theory solely to such inferences, this will clearly be insufficient to account for the conceptual of "bird" entirely in inferential terms. After all, it clearly seems essential to the meaning of "bird" that birds fly. That is, in general, if something's a bird, then it flies. Thus, in our inferential articulation of "bird," we should also include inferences such as the one from "Bella's a bird" to "Bella flies." Unlike the inference from "Bella's a bird" to "Bella's an animal," the inference from "Bella's a bird" to "Bella flies" is *defeasible* in that the addition of sentences to the premise set may function to *defeat* the goodness of this inference. For instance, if we add "Bella's a penguin," then this inference is no longer any good. Once one acknowledges that defeasible inferences of this sort are partly constitutive of the contents of *some* material concepts, it's hard to avoid the conclusion that they are partly constitutive of *nearly all* material concepts. For instance, when one actually spells out an inferentialist semantics for content words like "red," it will not suffice to inferentially connect this term with other color terms; in order to account for the distinctive conceptual significance of the color *red*, one will also have to countenance such inferences as "The flower is a rose" to the "The flower is red." However, this inference is defeasible as well. For instance, adding "The flower is a white rose" defeats it.

In the context of a sequent calculus of the sort made sense of by Restall-style bilateralism, there's a straightforward way to extend an approach for logical vocabulary to account for the non-logical meanings of atomic sentences: we simply include *non-logical axioms*. That is, in addition to having logical axioms of the form $A \vdash A$, we'll also include "material axioms" such as **red** $\vdash$ **colored**.[18] Such material axioms can be taken to be constitutive of the meanings of the atomic sentences they relate. So, if we want to inferentially account for such sentences as "Bella's a bird" and "Bella flies," we'll want to have **bird** $\vdash$ **flies** as a non-logical material "axiom" in our proof system. However, we won't want to have **bird**, **penguin** $\vdash$ **flies**. Accordingly, we'll have to reject the structural rule of *Weakening*:

$$\frac{X \vdash A, Y}{X, B \vdash A, Y}\ \text{Weakening}$$

One important feature of sequent calculi of the sort interpreted by Restall's bilateralism is that the use of structural rules such as Weakening themselves constitute logical steps in the use of proof system, rather than such rules being built into to structure of the proof system. This makes sequent calculi the natural setting for constructing substructural logics: logical systems that work without the use of such rules. Now, there are different reasons to want a logical system that works without such rules, but, in this inferentialist context, the reason is so that the system is able to accommodate sequents for which they actually fail. In the context of Restall's bilateralism, the failure of Weakening just discussed can be understood in terms of the fact that asserting "Bella's a bird" and denying "Bella flies" constitutes a *curably* "out of bounds" position. That is, there is sort of normative tension between these two acts, but this tension can be *cured* by additional assertions or denials, such as the additional assertion of "Bella's a penguin."

A logical system can be understood as functioning to extend the set of inferential relations between atomic sentences, given by the material axioms, to inferential relations between sentences of arbitrary logical complexity. For instance, given suitable rules for disjunction, one can move from **red** $\vdash$ **colored** and **blue** $\vdash$ **colored** to **red** $\vee$ **blue** $\vdash$ **colored**. Now, Gentzen's LK, the sequent

[18] I'll take bolded words of this sort as symbols for sentences such as "*a* is red" and "*a* is colored."

system officially endorsed by both Restall and Ripley, not only requires Weakening in order to function, but the connective rules directly enforce it.[19] For instance, consider one of Gentzen's left conjunction rules:

$$\frac{X, A \vdash Y}{X, A \wedge B \vdash Y}\ L_{\wedge_1}$$

This would let us go from the fact that asserting "Bella's a bird" is incompatible with denying "Bella flies" to the fact that asserting "Bella's a bird and she's a penguin" is incompatible with denying "Bella flies," and that is precisely what we don't want to say. However, though Gentzen's LK has a problem accommodating defeasible incompatibilities, by tweaking the rules, we can avoid such consequences. In particular, the following sequent calculus, developed by Ketonen (1994), accommodates defeasible non-logical axioms without any problems:[20]

$$\frac{}{X, A \vdash A, Y}\ \text{Containment}$$

Where X, Y, and $\{A\}$ contain only atomics.

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \mathrm{L}_{\neg} \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \mathrm{R}_{\neg}$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \mathrm{L}_{\wedge} \qquad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\ \mathrm{R}_{\wedge}$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y}\ \mathrm{L}_{\vee} \qquad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\ \mathrm{R}_{\vee}$$

$$\frac{X \vdash A, Y \quad X, B \vdash Y}{X, A \rightarrow B \vdash Y}\ \mathrm{L}_{\rightarrow} \qquad \frac{X, A \vdash B, Y}{X \vdash A \rightarrow B, Y}\ \mathrm{R}_{\rightarrow}$$

Note that the axiom schema here is distinct from the more familiar axiom schema of *Reflexivity*: $A \vdash A$. Ketonen's axiom schema generalizes Reflexivity to allow for axioms in which additional formulas have been added in on the

[19] Well, the sequent system Ripley endorses is not LK *per se*, but LK without the rule of Cut.

[20] For discussion of the formal properties of this sequent calculus, see Negri and von Plato (2008). Notably, in addition to Weakening, Contraction is eliminable as well, but I'll ignore this fact here, as I am, for simplicity, treating sequents as relating sets.

left or right. This builds in all of the Weakening one needs for classical logic in the axioms, and so Weakening as a structural rule can be eliminated. Because Weakening is eliminable, this system permits the addition of non-logical material axioms for which Weakening actually fails, and, unlike Gentzen's rules, the rules of this system play nicely with such axioms. For instance, if you look at the conjunction rules, you'll see that we can no longer derive **bird** $\wedge$ **penguin** $\vdash$ **flies**, from the sequent **bird** $\vdash$ **flies**. We need the sequent **bird**, **penguin** $\vdash$ **flies** which we won't include as a material axiom, since it's not a good material inference.

The move to a Ketonen-style sequent calculus has recently been motivated on these grounds by Brandom (2018), Hlobil (2018), Kaplan (2017, 2022), and Hlobil and Brandom (2024), and this move is one that bilateralists such as Restall and Ripley ought to welcome. Restall (2016) has encouraged proof-theoretic accounts of "concepts beyond the core logical constants," and one substantive step in that direction is providing an account, in terms of defeasible incompatibility relations, of material concepts such as "bird" and "flies." Moreover, Ripley (2017) has explicitly likened the notion of incompatibility codified by multiple conclusion sequents, on Restall's bilateralist interpetation of them, to Brandom's notion of "material incompatibility," and has proposed bilateralism for natural language inferentialist semantics. Because very many of the material incompatibility relations that must be codified to have an adequate account of meaning in natural language are defeasible like the example above, the sequent calculus Ripley should endorse is not Gentzen's, but Ketonen's which can play nicely with a non-monotonic consequence relation. Notably, Ripley (2013) has rejected *Transitivity* as a way of dealing with the semantic paradoxes. However, from the perspective of inferentialist semantics for natural language, a stronger case for the rejection of Transitivity, which does not rely on semantic paradoxes, has to do with the tight connection between Transitivity and Monotonicity. Consider, for instance, that there's a simple way of concocting a failure of Transitivity out of any failure of Monotonicity that involves a general rule with exceptions. To take the same example, we'll presumably want to have **bird** $\vdash$ **flies** and **penguin** $\vdash$ **bird**, but not **penguin** $\vdash$ **flies**. Thus, we have to reject the following rule, which I'll call "Simple Transitivity":

$$\frac{X \vdash A \quad A \vdash B}{X \vdash B}\ \text{Simple Transitivity}$$

Moreover, consider the principle of *Cumulative* Transitivity:

$$\frac{X \vdash A \quad X, A \vdash B}{X \vdash B}\ \text{Cumulative Transitivity}$$

This principle is weaker than Simple Transitivity in a non-monotonic context, but even it has clear counter-examples when we consider defeasible reasoning. For instance, we presumably want **bird** $\vdash$ **flies** and **bird**, **flies** $\vdash$ ¬**penguin**, but not **bird** $\vdash$ ¬**penguin**.[21] Of course, Gentzen's LK is able to accommodate such failures of Transitivity—this is a consequence of Gentzen's Cut-Elimination theorem. However, the Ketonen system shown above is able to accommodate both Monotonicity and Transitivity failures in a unified way. Moreover, these failures make perfect sense on Restall's bilateral interpretation.

Whereas the multiple conclusion systems that are the target of Restall's bilateralism enable us to straightforwardly incorporate non-logical axioms encoding defeasible material inferential relations, it's not at all clear how we can do something similar in the sort of bilateral systems proposed by Rumfitt. The most straightforward way to incorporate material inferences into a natural deduction system would be with primitive inference rules like the following:

$$\frac{+\langle \mathbf{red} \rangle}{+\langle \mathbf{colored} \rangle} \qquad \qquad \frac{+\langle \mathbf{red} \rangle}{-\langle \mathbf{green} \rangle}$$

But, of course, if we add in defeasible inferential rules like the following

$$\frac{+\langle \mathbf{bird} \rangle}{+\langle \mathbf{flies} \rangle} \qquad \qquad \frac{+\langle \mathbf{penguin} \rangle}{+\langle \mathbf{bird} \rangle}$$

we'll be able to link up inferences to illicitly infer **flies** from **penguin**. To block such inferences, it seems that we need a sequent calculus (or something very much like one) in order to keep track of background premises. Still, even transposing a Rumfitt-style bilateral system into sequent notation, it's

[21]See Simonelli (2022) for an extended discussion of these examples, as they arise in the context of natural language indicative conditionals.

hard to see how to do without structural rules such as Weakening when it comes to logically extending a set of material axioms via the operational rules. Consider, for instance, the derivation of $+\langle \textbf{red} \vee \textbf{yellow} \rangle \vdash +\langle \neg\textbf{blue} \rangle$ from $+\langle \textbf{red} \rangle \vdash -\langle \textbf{blue} \rangle$ and $+\langle \textbf{yellow} \rangle \vdash -\langle \textbf{blue} \rangle$ in Rumfitt's system (transposed into sequent notation):

$$\dfrac{\dfrac{\dfrac{}{+\langle \mathbf{r} \vee \mathbf{y} \rangle \vdash +\langle \mathbf{r} \vee \mathbf{y} \rangle}\,\text{Reflex.} \quad \dfrac{+\langle \mathbf{r} \rangle \vdash -\langle \mathbf{b} \rangle}{+\langle \mathbf{r} \vee \mathbf{y} \rangle, +\langle \mathbf{r} \rangle \vdash -\langle \mathbf{b} \rangle}\,\text{Weak.} \quad \dfrac{+\langle \mathbf{y} \rangle \vdash -\langle \mathbf{b} \rangle}{+\langle \mathbf{r} \vee \mathbf{y} \rangle, +\langle \mathbf{y} \rangle \vdash -\langle \mathbf{b} \rangle}\,\text{Weak.}}{+\langle \mathbf{r} \vee \mathbf{y} \rangle \vdash -\langle \mathbf{b} \rangle}\,+\vee_E}{+\langle \mathbf{r} \vee \mathbf{y} \rangle \vdash +\langle \neg\mathbf{b} \rangle}\,+\neg_I$$

Moreover, the operational rules of Rumfitt's system enforce the structural rule of Weakening in just the way those of Gentzen's LK do. Consider, for instance, the negative conjunction rules proposed by Rumfitt, exploiting the duality of conjunction and disjunction:

$$\dfrac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\,-\wedge_{\mathrm{I}_1} \qquad\qquad \dfrac{\Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\,-\wedge_{\mathrm{I}_2}$$

In this context, we can see that there is clearly a problem with these negative conjunction rules. For instance, let Γ be an assertion of "Sadie lays eggs," *A* be "Sadie's a mammal," and *B* be "Sadie's a platypus." Asserting "Sadie lays eggs," in general, commits one to denying "Sadie's a mammal," but it doesn't commit one to denying "Sadie's a mammal and she's a platypus." On the contrary, saying "Sadie's a mammal and she's a platypus" is perfectly compatible with asserting "Sadie lays eggs."

Now, one might think that the conjunction introduction rules can be modified to avoid this problem, and, indeed, as we'll soon see, they can be. However, the more fundamental problem for Rumfitt-style systems is that the very structure of natural deduction systems, containing both introduction and elimination rules, ends up amounting to an imposition of structural rules. To see this, consider the positive conditional rules, which are a paradigm component of Rumfitt-style bilateral natural deduction:

$$\frac{\Gamma, +\langle A\rangle \vdash +\langle B\rangle}{\Gamma \vdash +\langle A \to B\rangle}\; + \to_I \qquad\qquad \frac{\Gamma \vdash +\langle A \to B\rangle \quad \Gamma \vdash +\langle A\rangle}{\Gamma \vdash +\langle B\rangle}\; + \to_E$$

And recall again the failures of Cumulative Transitivity that arise in a non-monotonic context mentioned above. Putting the same example above in terms of commitments to asserting, intuitively, asserting "Bella's a bird" (defeasibly) commits one to asserting "Bella flies," and asserting "Bella's a bird" along with asserting "Bella flies" commits one to asserting "Bella's not a penguin," but asserting "Bella's a bird" does not, by itself, commit one to asserting "Bella's not a penguin." Given the introduction and elimination rules conditional rules, Cumulative Transitivity can be derived as follows:

$$\frac{\dfrac{\Gamma, +\langle A\rangle \vdash +\langle B\rangle}{\Gamma \vdash +\langle A \to B\rangle}\; + \to_I \qquad \Gamma \vdash +\langle A\rangle}{\Gamma \vdash +\langle B\rangle}\; + \to_E$$

One might think, at this point, that, to preserve a standard conditional, perhaps Cumulative Transitivity should just be accepted, and the bullet should be bit with regard to the intuitive failure just stated. However, as Hlobil (2016, 103) has shown, given a standard conditional that allows for Deduction ($\to_I$) and Detachment, if one's proof system contains an axiom schema of Containment, then Monotonicity *follows from* Cumulative Transitivity:[22]

$$\frac{\dfrac{\dfrac{}{\Gamma, +\langle A\rangle, +\langle B\rangle \vdash +\langle A\rangle}\;\text{Containment}}{\Gamma, +\langle A\rangle \vdash +\langle B \to A\rangle}\;\text{Deduction} \qquad \Gamma \vdash +\langle A\rangle}{\dfrac{\Gamma \vdash +\langle A \to B\rangle}{\Gamma, +\langle A\rangle \vdash +\langle B\rangle}\;\text{Detachment}}\;\text{CT}$$

Thus, insofar as one wants one's system to accommodate defeasible inferential relations, the sort of natural deduction systems developed by Rumfitt-style bilateralists face serious difficulties. Of course, I don't want to claim here that it's *impossible* for such systems to accommodate defeasible inferential relations.[23]

[22] I consider here the positively signed versions of Cumulative Transitivity, but, given the negation rules, this proof can be straightforwardly generalized to the full bilateral versions of these rules.

[23] For instance, in response to the derivation of Cumulative Transitivity with the conditional rules above, one might try to follow Tennant (1987) in restricting proofs to normal forms, thus ruling out this proof, since it involves an application $\to_I$ followed by an application of $\to_E$. However, such a restriction is hard to motivate in this context.

The point is just that defeasible inferential relations can be straightforwardly accommodated in a multiple conclusion sequent setting in the way shown above, providing serious motivation for adopting a Restall-style approach to inferentialism, as Hlobil and Brandom (2024) do.

So, in the context of providing an inferentialist semantics, both forms of bilateralisms have benefits, but both have serious limitations. On the one hand, there is a fundamental conceptual reason to favor a Rumfitt-style approach in articulating an inferentialist conception of meaning, as only such a conception affords us with genuine inferential rules. On the other hand, when it comes to actually carrying out an inferentialist theory of meaning, which will essentially involve incorporating defeasible inferential relations, the concrete logical systems readily made sense of by Restall-style bilateralism seem better-suited to the task. There is reason, then, to wonder if we can bring both forms of bilateralism together so as to get the best of both worlds. We can, and I'll now show how.

3 A Bridge Between Bilateralisms

Let us start by considering the multiple conclusion sequent calculus, on its standard unilateral interpretation. In such a calculus, we don't just have single and multiple conclusion sequents of the form $X \vdash A$ and $X \vdash Y$, but we also have sequents with an empty-righthand side of the form $X \vdash$, which are understood as expressing that the set of sentences in X are *jointly incoherent.* Insofar as a solely left-sided sequent expresses incoherence in a unilateral context, it is reasonable to think, then, that in a bilateral sequent calculus, such a sequent can express the same thing. That is, where Γ is a set of signed sentences, $\Gamma \vdash$ says that the set of stances in Γ, be they assertions or denials, are jointly incoherent. This suggests the straightforward translation of Restall's bilateralism into the signed notation proposed by Rumfitt. Recall, for Restall, a sequent of the form $X \vdash Y$ says that the position consisting in asserting everything in X and denying everything in Y is incoherent. To translate an unsigned multiple conclusion sequent of the form $X \vdash Y$, on Restall's interpretation, to a signed sequent of the form $\Gamma \vdash$ let $\Gamma = \{+\langle A\rangle \mid A \in X\} \cup \{-\langle B\rangle \mid B \in Y\}$. Conversely, to translate a signed sequent of the form $\Gamma \vdash$ to unsigned multiple conclusion sequent of the form $X \vdash Y$ let $X = \{A \mid +\langle A\rangle \in \Gamma\}$ and $Y = \{B \mid -\langle B\rangle \in \Gamma\}$. This is a faithful one-to-one translation, and so whole sequent systems can be translated in this manner. Thus, we can reformulate Ketonen's sequent calculus, shown above in its standard multiple conclusion presentation, in bilateral notation as follows:

$$\frac{}{\Gamma, \varphi, \varphi^* \vdash}\ \text{Incoherence}$$

Where Γ and $\{\varphi\}$ contain only signed atomics.

$$\frac{\Gamma, -\langle A\rangle \vdash}{\Gamma, +\langle \neg A\rangle \vdash}\ +_\neg \qquad \frac{\Gamma, +\langle A\rangle \vdash}{\Gamma, -\langle \neg A\rangle \vdash}\ -_\neg$$

$$\frac{\Gamma, +\langle A\rangle, +\langle B\rangle \vdash}{\Gamma, +\langle A \wedge B\rangle \vdash}\ +_\wedge \qquad \frac{\Gamma, -\langle A\rangle \vdash \quad \Gamma, -\langle B\rangle \vdash}{\Gamma, -\langle A \wedge B\rangle \vdash}\ -_\wedge$$

$$\frac{\Gamma, +\langle A\rangle \vdash \quad \Gamma, +\langle B\rangle \vdash}{\Gamma, +\langle A \vee B\rangle \vdash}\ +_\vee \qquad \frac{\Gamma, -\langle A\rangle, -\langle B\rangle \vdash}{\Gamma, -\langle A \vee B\rangle \vdash}\ -_\vee$$

$$\frac{\Gamma, -\langle A\rangle \vdash \quad \Gamma, +\langle B\rangle \vdash}{\Gamma, +\langle A \to B\rangle \vdash}\ +_\to \qquad \frac{\Gamma, +\langle A\rangle, -\langle B\rangle \vdash}{\Gamma, -\langle A \to B\rangle \vdash}\ -_\to$$

These rules with signed formulas now *show*, explicitly in the bilateral notation itself, exactly what the more familiar sequent rules *say*, on a bilateralist interpretation of them.

Having provided a faithful representation of the multiple conclusion sequent calculus in explicitly bilateral notation, let us now turn to the task of connecting this system to the sort of bilateral systems developed by Smiley and Rumfitt. Thus far, I have simply *stated* that a sequent of the form $\Gamma \vdash$ expresses the incoherence of the set of moves in Γ. This, of course, should not be simply stated, but, rather, should be *codified* in the rules of sequent system itself. Now, in a *uni*lateral system, the fact that $X \vdash$ encodes that the set of sentences in X are jointly incoherent can be understood in terms of the negation rules, which let one go from $X, A \vdash$ to $X \vdash \neg A$ and $X \vdash A$ to $X, \neg A \vdash$.[24] In a

[24] There are different ways of making sense of how such sequents encode incoherence. Perhaps the most standard way of understanding the fact is that $X \vdash$ encodes the incoherence of the sentences in X is in terms of the fact that, given Monotonicity, if you have such a sequent, you can conclude that $X \vdash A$, for any sentence A. Thus, $X \vdash$ can be understood as encoding the incoherence of X in terms of the fact that, from an incoherent set of sentences, anything follows. However, in this context, we've rejected Monotonicity. Accordingly, we should think of an alternative way of making sense of the fact that $X \vdash$ expresses the incoherence of all of the sentences in X.

*bi*lateral system, we can get the same behavior at the structural level by way of following pair of coordination principles:

$$\frac{\Gamma, \varphi \vdash}{\Gamma \vdash \varphi^{*}}\ \text{Out} \qquad\qquad \frac{\Gamma \vdash \varphi}{\Gamma, \varphi^{*} \vdash}\ \text{In}$$

The Out rule says that, if the position consisting in all of the stances in Γ along with stance φ is incoherent, then Γ commits one to taking the opposite stance φ^{*}, whereas the In rule says that, if Γ commits one to taking the stance φ, then the position consisting in Γ along with the opposite stance φ^{*} is incoherent.[25] With these rules in view, consider the following sequent:

$$+\langle \textbf{red} \rangle, +\langle \textbf{green} \rangle \vdash$$

This says that the position consisting in asserting "*a* is red" and asserting "*a* is green" is incoherent. The incoherence of the position consisting in both of these assertions can be understood in terms of the fact that asserting "*a* is red" commits one to denying "*a* is green" and asserting "*a* is green" commits one to denying "*a* is red." The relation between all of these incoherence and incompatibility facts is codified by In and Out, as, given these rules, this sequent is equivalent to this one:

$$+\langle \textbf{red} \rangle \vdash -\langle \textbf{green} \rangle$$

and this one:

$$+\langle \textbf{green} \rangle \vdash -\langle \textbf{red} \rangle$$

Whereas the sequent with both assertions on the left can be understood as encoding an *incoherence property* of that set of sentences, these sequents, with an assertion on the left and a denial on the right can be understood as encoding an *incompatibility relation* between the sentences.

[25] Note (if we treat the empty right-hand side as containing a "$\perp$"), Out is just the version of Reductio appealed to in standard Rumfitt-style bilateral approaches that split Smiliean Reductio into a pair of coordination principles. In, on the other hand, as not been explicitly considered in the context of Rumfitt-style bilateralism, since it is only natural in a sequent context.

In and Out together are equivalent the coordination principle that Smiley (1996) calls "Reversal," with an additional clause allowing for empty righthand sides:

$$\frac{\Gamma, A \vdash B}{\Gamma, B^{*} \vdash A^{*}} \text{ Reversal}$$

Where {A} or {B} can be null.

Given our explicitly bilateral translation of Restall's bilateralism, In and Out (or, equivalently, Reversal), in effect, constitute a *bridge between bilateralisms*, enabling us to move from (faithfully translated) multiple conclusion-sequents, encoding *incoherence*, to Rumfitt-style sequents, encoding *consequence*, and vice versa. Of course, the actual acceptance of such rules is sure to be controversial among many logicians, not least of whom will be Restall and Ripley, who want to resist a committive notion of logical consequence altogether.[26] Nevertheless, I take it that all parties should welcome a formal framework of this sort in which the moves that would bridge bilateralism are formally represented.[27] Regardless of whether one accepts In and Out as sound bilateral structural rules, one can systematically investigate various different possibilities for bilateralism of both sorts, with different bilateral structural rules and different operational rules, logically mapping the philosophical landscape.

There are a number of bilateral systems that can be considered in this unified bilateral framework. In what follows, I'll consider two such systems. In the next section, I'll consider a system that makes use of Reversal to meet

[26]I mentioned above Restall's resistance earlier, but of particular note here is Ripley's (2013) approach to the liar paradox that crucially hangs on the bilateral reading of the turnstile. Ripley maintains that the liar sentence λ is such that, relative to any position Γ, asserting λ is out of bounds and so is denying λ; however, that does not entail that any position Γ is itself out of bounds. Translating Ripley's approach in this explicitly bilateral notation, Ripley allows us to derive both $\Gamma, +\langle\lambda\rangle \vdash$ and $\Gamma, -\langle\lambda\rangle \vdash$, but he rejects the principle of *Extensibility* (unilateral Cut), which enables us to infer $\Gamma \vdash$ from Γ, φ and Γ, φ^{*} now, plausibly, though he wants to say that Γ along with denying λ is incoherent, he *doesn't* want to say that Γ commits one to asserting λ. So, he'd want to reject Out (though perhaps while accepting In).

[27]In this way, the multiple steps involved in this bridge between bilateralisms—which first translates multiple conclusion sequents exactly in a way that simply makes Restall's bilateral interpretation explicit, and then imposes bilateral structural rules to transform them into Rumfitt-style sequents—constitutes a decisive advantage over existing proposals for translating between multiple conclusion sequents into Rumfitt-style bilateral sequents, such as Hjortland's (2014, 444), which simply translates $\Gamma \vdash \Delta$ as $+\langle\gamma_1\rangle, +\langle\gamma_2\rangle \ldots +\langle\gamma_n\rangle, -\langle\delta_2\rangle \ldots -\langle\delta_n\rangle \vdash +\langle\delta_1\rangle$.

the desiderata for an inferentialist logic articulated in the previous section. In the section following next, I'll show how Reversal can be dropped to achieve a greater degree of substructural freedom than systems considered thus far, motivated by the same sort of considerations involving defeasible inferences considered in Section 2.

4 The Best of Both Worlds

Above, I directly translated Ketonen's multiple conclusion sequent calculus as relating bilateral sequents of the form $\Gamma \vdash$. It's now explicit in our notation that such a sequent calculus specifies only the conditions under which asserting or denying some logically complex sentence is *incoherent*, doing so in terms of the incoherence of other positions. As many proponents of proof-theoretic semantics have argued, however, it is crucial to proof-theoretically defining the meanings of the connectives that our proof system specify their meaning in terms of their *inferential* relations. In particular, in the tradition following Gentzen, the introduction rules are treated as having pride of place in defining the meaning of some connective, for such rules articulate the conditions under which a sentence containing that connective is to used. Thus, let us rewrite the sequent calculus itself so as to turn it from a calculus of *incoherence* to a calculus of *consequence*, specifying the conditions under which a sentence containing a given connective is to be asserted or denied:[28]

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \text{Containment}$$

Where Γ and $\{\varphi\}$ contain only signed atomics.

$$\frac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash +\langle \neg A \rangle}\ +_{\neg} \qquad \frac{\Gamma \vdash +\langle A \rangle}{\Gamma \vdash -\langle \neg A \rangle}\ -_{\neg}$$

$$\frac{\Gamma \vdash +\langle A \rangle \quad \Gamma \vdash +\langle B \rangle}{\Gamma \vdash +\langle A \wedge B \rangle}\ +_{\wedge} \qquad \frac{\Gamma, +\langle A \rangle \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\ -_{\wedge}$$

[28]There are different ways to do this. Simonelli (2024) takes the top sequents of the negative conjunction and positive disjunction rules to be $\Gamma, +\langle A \rangle, +\langle B \rangle \vdash$ and $\Gamma, -\langle A \rangle, -\langle B \rangle \vdash$ respectively. Since, in the context of Reversal, such sequents are immediately inter-provable with the top sequents shown here, little hangs on this decision.

$$\frac{\Gamma, -\langle A\rangle \vdash +\langle B\rangle}{\Gamma \vdash +\langle A \vee B\rangle} +_\vee \qquad \frac{\Gamma \vdash -\langle A\rangle \quad \Gamma \vdash -\langle B\rangle}{\Gamma \vdash -\langle A \vee B\rangle} -_\vee$$

$$\frac{\Gamma, +\langle A\rangle \vdash +\langle B\rangle}{\Gamma \vdash +\langle A \to B\rangle} +_\to \qquad \frac{\Gamma \vdash +\langle A\rangle \quad \Gamma \vdash -\langle B\rangle}{\Gamma \vdash -\langle A \to B\rangle} -_\to$$

Clearly, given Reversal, this system is equivalent to the previous one. However, unlike the previous system, which directly translates Ketonen's sequent calculus on Restall's interpretation of it, we now have a system that can be conceived of as specifying the conditions under which one is *committed* to asserting or denying a logically complex sentence. Indeed, the negation rules, the conditional rules, the positive conjunction rule, and the negative disjunction rule are the familiar introduction rules from Rumfitt's system. The less familiar ones are the negative conjunction and positive disjunction rules, though these have recently been proposed, for independent reasons, by del Valle-Inclan and Schlöder (2023) in the context of a Rumfitt-style natural deduction system. The negative conjunction rule says that if a set of stances Γ along with an assertion of A commits one to denying B, then Γ commits one to denying $A \wedge B$. Note that, given Reversal, if Γ along with an assertion of A commits one to denying B, then, just as well, Γ along with an assertion of B commits one to denying A, and both sequents can be understood in terms of the fact that Γ along with an assertion of B and an assertion of A is incoherent. So, essentially, this rule for conjunction says that you're committed to denying a conjunction just in case, given your stances, it's incoherent to assert both conjuncts such that asserting one of the conjuncts commits you to denying the other. Dually, the positive disjunction rule says that you're committed to asserting a disjunction just in case it's incoherent to deny both disjuncts such that denying one commits you to asserting the other. From a purely intuitive standpoint, I take this set of rules to be at least as good of a candidate for a proof-theoretic specification of the meanings of the logical connectives as the rules in the systems proposed by Smiley or Rumfitt. Technically, however, this system has all of the benefits of Ketonen's sequent calculus over standard Smiley/Rumfitt-style systems.

Like Ketonen's sequent calculus, this system requires neither Monotonicity nor Transitivity in order to function. Accordingly, we can use this system to logically extend material axioms without the use of structural rules like

Weakening or Smiliean Reductio. Consider again the example from Section 2 of deriving $+\langle \mathbf{red} \vee \mathbf{yellow} \rangle \vdash +\langle \neg \mathbf{blue} \rangle$ from $+\langle \mathbf{red} \rangle \vdash -\langle \mathbf{blue} \rangle$ and $+\langle \mathbf{yellow} \rangle \vdash -\langle \mathbf{blue} \rangle$. Recall, this required the use of Weakening in Rumfitt's system. In this system, no Weakening or Reductio is required:

$$\dfrac{\dfrac{\dfrac{\dfrac{+\langle \mathbf{r} \rangle \vdash -\langle \mathbf{b} \rangle}{+\langle \mathbf{b} \rangle \vdash -\langle \mathbf{r} \rangle}\,\text{RV} \quad \dfrac{+\langle \mathbf{y} \rangle \vdash -\langle \mathbf{b} \rangle}{+\langle \mathbf{b} \rangle \vdash -\langle \mathbf{y} \rangle}\,\text{RV}}{+\langle \mathbf{b} \rangle \vdash -\langle \mathbf{r} \vee \mathbf{y} \rangle}\,-_{\vee}}{+\langle \mathbf{r} \vee \mathbf{y} \rangle \vdash -\langle \mathbf{b} \rangle}\,\text{RV}}{+\langle \mathbf{r} \vee \mathbf{y} \rangle \vdash +\langle \neg \mathbf{b} \rangle}\,+_{\neg}$$

Moreover, consider how the negative conjunction rule avoids the problem faced by the standard pair of negative conjunction rules. Once again, the standard rules are as follows:

$$\dfrac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\,-\wedge_{\text{I}_1} \qquad\qquad \dfrac{\Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\,-\wedge_{\text{I}_2}$$

Recall, the problem is that, though asserting to "Sadie's a mammal" commits one to denying "Sadie lays eggs," asserting "Sadie's a mammal" doesn't commit one to denying "Sadie's a platypus and she lays eggs." This problem is avoided with our single negative conjunction rule:

$$\dfrac{\Gamma, +\langle \varphi \rangle \vdash -\langle \psi \rangle}{\Gamma \vdash -\langle \varphi \wedge \psi \rangle}\,-_{\wedge}$$

This rule precludes one from being able to derive the problematic sequent $+\langle \textbf{mammal} \rangle \vdash -\langle \textbf{platypus} \wedge \textbf{lays eggs} \rangle$, since, though one *will* have the material axiom $+\langle \textbf{mammal} \rangle \vdash -\langle \textbf{lays eggs} \rangle$, one *won't* have the material axiom $+\langle \textbf{mammal} \rangle, +\langle \textbf{platypus} \rangle \vdash -\langle \textbf{lays eggs} \rangle$, which is what one needs in order to apply this negative conjunction rule and derive the problematic sequent. In this way, this bilateral sequent calculus enables us to take a set of base sequents, encoding defeasible inferential relations such as $+\langle \textbf{bird} \rangle \vdash +\langle \textbf{flies} \rangle$ and $+\langle \textbf{mammal} \rangle \vdash -\langle \textbf{lays eggs} \rangle$, and expand them to sequents relating uses of logically complex vocabulary, thereby determining the discursive significance of such uses.

So, we get all the benefits of using Ketonen's multiple-conclusion sequent calculus articulated above, and we retain Restall's interpretation of it if we consider only the solely left-sided fragment of the consequence relation. However, we now have a turnstile expressing a relation of *consequence*, properly so-called, between sets of assertions and denials and single assertions or denials. Accordingly, we can think of sequents with formulas on the right-hand side as articulating the properly *inferential* significance of the sentences they contain, specifying rules for attributing commitments to assertions or denials to speakers on the basis of the assertions and denials they've made. Bringing together these two bilateralisms in the way that we have, we are able to get the best of both worlds.

5 A Further Degree of Substructural Freedom

The bilateral system I have laid out meets the above stated desiderata in being, on the one hand, a single-conclusion Rumfitt-style system capable of specifying the conditions under which sentences are to be used, and, on other hand, equivalent to Ketonen's multiple conclusion sequent calculus and so able to accommodate defeasible material inferences in just the same way. However, given that the system I've laid out is equivalent to Ketonen's, one might think that, while system delivers single conclusion sequents which have the philosophical advantages articulated above, as far as the *logic* is concerned, it is simply Ketonen's multiple conclusion sequent calculus in disguise. Indeed, given everything I've said so far, one might make a division between an incoherence-based *semantics*, given in terms of a multiple conclusion sequent calculus understood on Restall's bilateral interpretation, and an inferential *pragmatics* which one obtains, in a second step, by signing sequents and applying Out to them in the way shown above. Though Hlobil and Brandom (2024) do not explicitly put forward a bilateral system that enables the inferential pragmatics to be formally in specified in this way, this is essentially the approach that they adopt. In the formal proof-theoretic semantics, they propose a multiple conclusion sequent calculus, interpreted *a la* Restall-style bilateralism, and then, in the informal explication of how the system fits into an inferentialist approach to meaning, they endorse the principle that if one is precluded from being entitled to deny A, then one is implicitly committed to asserting A, and likewise if one is precluded from being entitled to assert A, then one is implicitly committed to denying A (46-48). The bilateral framework that I've put forward makes this approach formally explicit in terms of the translation of Restall-style bilateralism into solely left-sided bilateral sequents and the coordination principles of In and Out. What I now want to show is how this new formal representation opens up a new degree of substructural freedom of just the sort that Hlobil and Brandom motivate.

We saw in Section 2 that, by explicitly representing the use of structural rules such as Weakening or Transitivity as logical steps, a sequent framework is naturally suited to considering systems in which such structural rules fail. We can now apply the same lesson in the context of an explicitly bilateral framework. That is, by explicitly representing the coordination principle of Reversal as a logical move, this framework enables us to consider systems in which Reversal fails. There are a number of reasons why Reversal might be questioned. Sticking with the main philosophical motivation of concern in this paper, I want to consider reasons having to do with incorporating defeasible material inferences into an inferentialist framework. For a concrete case, consider that asserting "Sadie's a mammal" defeasibly commits one to denying "Sadie lays eggs," and this committive consequence relation surely isn't defeated by a denial of "The moon is made of cheese." Thus, we have:

$$+\langle \mathbf{mammal}(s)\rangle, -\langle \mathbf{cheese}(m)\rangle \vdash -\langle \mathbf{lays\ eggs}(s)\rangle$$

Though a single application of Reversal, we get:

$$+\langle \mathbf{mammal}(s)\rangle, +\langle \mathbf{lays\ eggs}(s)\rangle \vdash +\langle \mathbf{cheese}(m)\rangle$$

Thus, asserting "Sadie's a mammal" and asserting "Sadie lays eggs" commits one to asserting "The moon is made of cheese." But this sort of explosion seems problematic, especially insofar as asserting "Sadie's a mammal" and asserting "Sadie lays eggs" amounts to making a set of moves that are only *defeasibly* incoherent. The core issue here is that Reversal essentially amounts to treating *incompatibilities between* assertions and denials in terms of the *incoherence of sets* of assertions and denials. But, as this example shows, it's plausible to think that we might have an incoherent set of assertions and denials Γ without it being the case that, for any $\varphi \in \Gamma$, we have $\Gamma/\{\varphi\} \vdash \varphi^*$.

A related case having to do with defeasible inferences can further motivate the rejection of Reversal. One might reasonably think that, even though asserting "Sadie's a mammal" defeasibly commits one to *denying* "Sadie lays eggs," asserting "Sadie's a mammal" along with *asserting* "Sadie lays eggs" *doesn't* commit one denying "Sadie lays eggs." In general, it seems reasonable to think that a defeasible implication to some stance can, at least in some cases, be defeated by the explicit adoption of the opposite of the stance to which one is defeasibly committed.[29] So, we might plausibly wish to reject the following principle, which I'll call Minimal Persistence:

$$\frac{\Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash \varphi}\ \text{Minimal Persistence}$$

However, given Expansion, Minimal Persistence follows from Reversal:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash}\ \text{In}}{\Gamma, \varphi^*, \varphi^* \vdash}\ \text{Expan.}}{\Gamma, \varphi^* \vdash \varphi}\ \text{Out}$$

Since Expansion seems to be non-negotiable in this context, if one wants to reject Minimal Persistence to accommodate this sort of defeat, one must reject Reversal.

Of course, one might object to the above reasons for rejecting Reversal, maintaining the simple sequent system proposed above. The main point is to show that, in this unified bilateral context, it is straightforward to adapt the above system so that Reversal is not imposed. We simply apply Reversal to the rules to as to obtain corresponding left rules, and this enables us to remove Reversal as a structural rule. Doing this, we obtain the following calculus:

[29]Indeed, this general sort of principle (though not formulated in quite these terms) is standard in many default logics following Reiter (1980).

$$\frac{}{\Gamma, \varphi \vdash \varphi}\ \text{Containment} \qquad \frac{}{\Gamma, \varphi, \varphi^* \vdash \psi}\ \text{Bilateral Explosion}$$

$$\frac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash +\langle \neg A \rangle}\ +_{\neg R} \qquad \frac{\Gamma \vdash +\langle A \rangle}{\Gamma \vdash -\langle \neg A \rangle}\ -_{\neg R} \qquad \frac{\Gamma, -\langle A \rangle \vdash \varphi}{\Gamma, +\langle \neg A \rangle \vdash \varphi}\ +_{\neg L} \qquad \frac{\Gamma, +\langle A \rangle \vdash \varphi}{\Gamma, -\langle \neg A \rangle \vdash \varphi}\ -_{\neg L}$$

$$\frac{\Gamma \vdash +\langle A \rangle \quad \Gamma \vdash +\langle B \rangle}{\Gamma \vdash +\langle A \wedge B \rangle}\ +_{\wedge R} \qquad \frac{\Gamma, +\langle A \rangle \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\ -_{\wedge R_1} \qquad \frac{\Gamma, +\langle B \rangle \vdash -\langle A \rangle}{\Gamma \vdash -\langle A \wedge B \rangle}\ -_{\wedge R_2}$$

$$\frac{\Gamma, +\langle A \rangle, +\langle B \rangle \vdash \varphi}{\Gamma, +\langle A \wedge B \rangle \vdash \varphi}\ +_{\wedge L} \qquad \frac{\Gamma, -\langle A \rangle \vdash \varphi \quad \Gamma, -\langle B \rangle \vdash \varphi}{\Gamma, -\langle A \wedge B \rangle \vdash \varphi}\ -_{\wedge L}$$

$$\frac{\Gamma, -\langle A \rangle \vdash +\langle B \rangle}{\Gamma \vdash +\langle A \vee B \rangle}\ +_{\vee R_1} \qquad \frac{\Gamma, -\langle B \rangle \vdash +\langle A \rangle}{\Gamma \vdash +\langle A \vee B \rangle}\ +_{\vee R_2} \qquad \frac{\Gamma, +\langle A \rangle \vdash \varphi \quad \Gamma, +\langle B \rangle \vdash \varphi}{\Gamma, +\langle A \vee B \rangle \vdash \varphi}\ +_{\vee L}$$

$$\frac{\Gamma, -\langle A \rangle, -\langle B \rangle \vdash \varphi}{\Gamma, -\langle A \vee B \rangle \vdash \varphi}\ -_{\vee L} \qquad \frac{\Gamma \vdash -\langle A \rangle \quad \Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \vee B \rangle}\ -_{\vee R}$$

$$\frac{\Gamma, +\langle A \rangle \vdash +\langle B \rangle}{\Gamma \vdash +\langle A \to B \rangle}\ +_{\to R_1} \qquad \frac{\Gamma, -\langle B \rangle \vdash -\langle A \rangle}{\Gamma \vdash +\langle A \to B \rangle}\ +_{\to R_2} \qquad \frac{\Gamma, -\langle A \rangle \vdash \varphi \quad \Gamma, +\langle B \rangle \vdash \varphi}{\Gamma, +\langle A \to B \rangle \vdash \varphi}\ +_{\to L}$$

$$\frac{\Gamma \vdash +\langle A \rangle \quad \Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \to B \rangle}\ -_{\to R} \qquad \frac{\Gamma, +\langle A \rangle, -\langle B \rangle \vdash \varphi}{\Gamma, -\langle A \to B \rangle \vdash \varphi}\ -_{\to L}$$

where $\{\varphi\}$ may be empty in the left rules and $\{\psi\}$ may be empty in Bilateral Explosion.

Whereas the bilateral system proposed in the previous section was *equivalent to* Ketonen's sequent calculus, this system is not equivalent to but *contains* Ketonen's sequent calculus, as we translated it in Section 3, as its solely left-sided fragment. In this way, it treats both *incoherence* and *consequence* without taking either to be reducible to the other. Of course, if we don't include material axioms, this logic generates just the same consequence relation as the previous system, namely, that of bilateral classical logic.[30] However, if we include material axioms, the two consequence relations may come apart. In particular, this system straightforwardly accommodates the examples of failures of Reversal considered above.

[30]Though I don't provide a proof of this claim here, it's perhaps worth noting that if one omits Bilateral Explosion, one obtains a bilateral sequent system for the paraconsistent logic LP proposed by Priest (1979).

6 Conclusion

Until now, bilateralist developments of inferentialism have largely proceeded *separately*, being pursued either in a Rumfitt-style fashion or in a Restall-style fashion. I have provided here a unified formal framework capable of representing both forms of bilateralism, with solely left-sided sequents expressing *incoherence*, the core notion of Restall-style bilateralism, and sequents that have a formula on the righthand side expressing *consequence*, the core notion of Rumfitt-style bilateralism. This enables debates between bilateralists, which have hitherto proceeded on informal grounds, to be represented as debates over whether to impose specific formal principles such as In and Out. Moreover, as I've shown, this enables the formulation of systems that combine the virtues of the systems advocated proponents of the respective versions of bilateralism and, in fact, go beyond them.

Technical Appendix

Definition 1: We'll refer to the system presented in Section 4 (containing Reversal) as "BK1," and the system presented in Section 5 (with both left and right rules) as "BK2."

Proposition 1: BK1 is equivalent to Ketonen's sequent calculus K.

Proof: I'll just sketch the proof strategy—it's easy (though a bit tedious) to fill in the details.[31] Note first that, given the translation procedure, the solely right-sided bilateral sequent calculus in Section 3 is a notational variant of K. Let us call this K'. We now establish that any sequent of the form $\Gamma \vdash$ in K' corresponds to an equivalence class of BK sequents under Reversal of the form $\Gamma/\{\varphi\} \vdash \varphi^*$ for any $\varphi \in \Gamma$, and we on proof height to show that this correspondence is preserved across proofs in the two systems (where applications of Reversal in BK are not taken to contribute to proof height). For the base case, we show that the equivalence holds for axioms. For the inductive step, we assume that the correspondence holds up to proof height n, and show that it holds at height $n + 1$ for each of the connective rules. □

[31] See Simonelli (2024) for the details and various other meta-theoretic results about BK1.

Definition 2: A bilateral sequent $\Gamma \vdash \varphi$ (where $\{\varphi\}$ may be null) is *classically valid*, $\Gamma \vDash_{CL} \varphi$ just in case there is no classical valuation v such that all positively signed sentences in Γ are true, all of the negatively signed sentences in Γ are false, and $\{\varphi\}$ is null or the sentence signed in the formula φ is false if φ is positively signed or true if φ is negatively signed.

Proposition 2: BK1 proves $\Gamma \vdash \varphi$ just in case $\Gamma \vDash_{CL} \varphi$.

Proof: It follows from the completeness of K and our translation schema that, for any classical validity of the form $\Gamma \vDash_{CL}$, K′ derives $\Gamma \vdash$. Accordingly, since, $\Gamma \vDash_{CL} \varphi$ is just in case $\Gamma, \varphi^* \vDash_{CL}$, for any such inference, K′ derives $\Gamma, \varphi^* \vdash$, and so, by Proposition 1, BK1 derives $\Gamma \vdash \varphi$. □

Proposition 3: BK2 proves $\Gamma \vdash \varphi$ just in case $\Gamma \vDash_{CL} \varphi$.

Proof: Given Proposition 2, the proof is straightforward. For the left to right direction, it is obvious that all axioms and rules of BK2 are derivable in BK1 through a single application of Reversal. For the right to left direction, it is straightforward to show by induction on proof height that Reversal is admissible in BK2, and thus, since BK2 contains all of the rules of BK1 (which is complete), it is also complete. □ .

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