Meaning, Coherence, and Consequence
Rethinking the Philosophical Significance of the (Classical) Sequent Calculus

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0 Introduction

In the *Philosophical Investigations*, Wittgenstein tells us that, for a large class of uses of the word "meaning," "the meaning of a word is its use in the language," (*PI*, § 43). Now, Wittgenstein himself clearly did not intend this claim as a substantive philosophical thesis. He meant it, rather, as a nothing more than a clarification about our use of the term "meaning," the simple observation that, when we inquire about the meaning of some word, our inquiry is generally settled by a specification of that word's use, where this specification can take a number of forms whose aptness will depend on the context in which the question of meaning has been raised. There are exetegical disputes one might enter into as to what, exactly, is meant by Wittgensten's remark in the context of the *Philosophical Investigations*, but, however one wants to interpret it, it's clear from the context of this remark that Wittgenstein did not intend to be initiating a philosophical program with it. Yet, despite his own intentions in saying what he did, this remark has been turned into a slogan for a philosophical program for understanding meaning in general: *meaning is use*. Here is Michael Kremer's take on this slogan, from his 1988 paper "Kripke and the Logic of Truth":

> "If we take the slogan 'meaning is use' seriously, we will be led to think of a language in quite a different way [than the standard

> model-theoretic conception]: roughly, as a syntactic structure together with a set of rules of use. To interpret a language, on this picture, is to assign a set of rules of use to each term of the language," (1988b, 270).

Kremer hereby expresses a program, driven by the slogan "meaning is use," of systematically assigning rules of use to the expressions of a langauge, and, in doing so, accounting for their linguistic meaning.

The person who has done the most work in the attempt to actually carry out this program over the past four decades was one of Kremer's teachers at Pittsburgh, Robert Brandom. Indeed, Brandom's work, like that of Dummett before him, might be summed up as an attempt to take the slogan "meaning is use" seriously. The first sentence of the Précis of Brandom's magnum opus *Making It Explicit*, for instance, reads, "The book is an attempt to explain the *meanings* of linguistic expressions in terms of their use." Crucially, for Brandom, unlike Wittgenstein, not all aspects of the use of linguistic expressions are on equal theoretical footing. Brandom assigns theoretical pride of place to the use of linguistic expressions in *inference*. Following Frege's context principle, Brandom understands the meaning of a word in terms of its role in *sentences* and, following an idea that he finds in Frege's *Begriffschrift*, Brandom understands meaning of a sentence in terms of its role in *inferences*, both inferences *to* that sentence from other sentences and inferences *from* that sentence to other sentences.[1] Brandom thus coins the term "inferentialism" to denote the particular use-theory of meaning according to which the sole aspect of a sentence's use that one appeals to in accounting for its meaning is its *inferential* use.

The inferentialist thesis that for every expression belonging to the lexicon of a natural language, the entirety of its meaning can be spelled out solely in terms of its inferential role is an ambitious one, to put it mildly. There are many aspects of the linguistic meaning of at least some expressions that it seems difficult if not impossible to account for in entirely intra-linguistic terms—in terms of inferences from sentences to other sentences. For instance, the use of words expressing empirical concepts like "red" and "cardinal" seems to be essentially linked, not just to the use of other words and sentences, but

[1] See Brandom (1983) for this geneology of his position.

to such things as *the color red* and *cardinals*. That is to say, their use seems to essentially involve not just an *inferential* dimension, but a *representational* dimension as well. In his 2010 paper, "Representation or Inference: Must We Choose? Should We?" Kremer argues, against Brandom, that inference should not be given pride of place over representation—that we should recognize linguistic meaning as essentially involving both a inferential dimension *and* a representational dimension, and one is not to be privileged over the other. I have tried to elsewhere (Simonelli 2023) to defend the global inferentialist position in response to Kremer's criticisms. I do think I can make a case that the position is not quite as crazy as it seems, but I'm not going to pursue this line of thought here. Instead I want to consider Kremer's own defense of inferentialism, much earlier in his philosophical career, for one restricted class of linguistic expressions.

A critical test case for the inferentialist account of meaning is the case of *logical vocabulary*, words like "not," "and," "or," and "if . . . then . . . ." Whether or not inferentialism is a plausible thesis for language in general remains to be seen, but one negative thesis regarding inferentialism is clear: if inferentialism cannot be made to work for specifically logical vocabuarly, then inferentialism for language in general is surely hopeless. Unlike the meanings of empirical words like "red" and "cardinal," the meanings of logical words like "not" and "and" don't seem to have an essentially representational dimension. Rather, it seems perfectly plausible on its face that, as Kremer puts it, "to understand the meaning of a piece of logical vocabulary is precisely to understand how it functions in inference," (1988b, 270). So inferentialism, for logical vocabualry, is intuitively plausible. Moreover, in terms of the systematic development of inferentialism, it seems that, here, inferentialists already have much of the work done for them: for existing proof systems seem to articulate just the sorts of rules in terms of which the inferentialist thinks the meanings of the logical connectives can be understood. In particular, the proof systems developed by Gerhard Gentzen seem particularly suited to the task. In this paper, I'm going to consider Kremer's attempt to articulate how one of Gentzen's proof systems—the sequent calculus—can be understood as providing an inferentialist account of the meanings of the logical connectives.

Here's the plan. In Section One, I explain the sequent calculus and Kremer's proposal—which goes back to his 1986 dissertation—for understanding its philosophical significance in terms of the fact that it provides an inferentialist account of th emeanings of the logical connectives. In Section Two, I provide very brief history of Brandom's inferentialism over the past four decades, and explain how it is only very recently that he has caught up to Michael and decided to do things in terms of the sequent calculus. This will also provide some further motivation for the sequent calculus formalization of inferentialism. In Section Three, I raise a fundamental problem for the understanding the meanings of the classical connectives in terms of the sequent calculus: the classical sequent calculus essentially involves sequents with *multiple conclusions*, and it's not at all clear how such sequents are to be understood. In Section Four, I introduce a conception of the multiple conclusion sequent calculus that has gained prominence in recent years in response to this issue: the *bilateralist* conception, according to which logic ends up not being about consequence at all but about the coherence or incoherence of sets of affirmations and denials. While this resolves the problem of making sense of multiple conclusion sequents, it does so only at the cost of eliminating the very notion of something's following from something else from our conception of logic. Finally, in Section Five, I propose a new kind of sequent calculus, which is equivalent to the classical sequent calculus, but which resolves both the conceptual problems with multiple conclusions while also enabling us to retain the idea that consequence is one of logic's principal concerns.

1 Kremer on the Significance of the Sequent Calclus

In his 1935 dissertation, Gentzen introduced two new kinds of proof stystems: *natural deduction* and the *sequent calculus*. The first sort of system is the one that is nowadays taught in most introductory logic courses. Beyond its use in introductory logic, however, natural deduction is the principal sort of system appealed to by philosophers interested in providing an inferentialist account of the meanings of the logic connectives. Gentzen's second sort of system—the sequent calculus—while it has a small circle of hardcore proponents, is still not widely appreciated in broader philosophical circles.[2] The key difference between these two systems is that, whereas natural deduction systems provide rules for moving from *particular formulas* to other particular formula, sequent systems provide rules for moving from *sequents*, which themselves encode inferential relations between particular formulas, to other sequents. So, whereas one operates at the *first-order inferential* level in using a natural deduction system, one operates at the *meta-inferential* level in using a sequent calculus. In his 1988 paper, "Logic and Meaning: The Philosophical Significance of the Sequent Calculus," Kremer explores the idea that the sequent calculus can be understood as providing an inferentialist account of the meanings of the logical connectives.

Now, as I noted, the classical sequent calclus notably features multiple conclusions, and I will get to this fact and the issues it raises shortly. First, however, to explain Kremer's basic proposal for understanding the significance of the sequent calculus, let us consider the more intuitive single conclusion case. Here, a sequent of the form $\Gamma \vdash A$, where Γ is a set of sentences and A is a single sentence, can be understood as saying that the conclusion A follows from the premises Γ, that Γ implies A. Unlike a natural deduction system, where the connectives are given introduction and elimination rules, in the sequent calculus, logical connectives are given just introduction rules: a rule (or number of rules) for introducing a sentence with that connective on the *left* side of a sequent, and a rule (or number of rules) for introducing a sentence with that connective on the *right* side of a sequent. Consider, for instance, the following left and right rules for conjunction:

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \mathrm{L}_{\wedge} \qquad\qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \mathrm{R}_{\wedge}$$

The left rule says that if some conclusion C follows from A along with B (along with some set of auxillary premises Γ), then C follows from Γ along with $A \wedge B$. The right rule says that if some set of premises Γ implies A and Γ also implies B, then Γ implies $A \wedge B$.

[2] Gentzen himself regarded the sequent calclus as a little more than a convenient formal set-up—much more artificial than the natural deduction set-up—in the context of which he could proof his main theorem of Cut Elimination.

Kremer's key thought in interpreting such rules as providing an inferentialist semantics for the logical connectives is that the inferential role of a sentence can be understood in terms of two aspects: its role as a *premise* of inferences—where various other things *follow from it* (potentially along with other things)—and its role as a *conclusion* of inferences—where *it follows from* various other things. These two aspects are respectively codified by the left and right rules of a sequent calculus. Kremer writes:

> "[T]o grasp the meaning of a logical constant is just to know how it behaves in inference. We can think of the sequent calculus introduction rules for a logical constant as representing the two aspects of its use; the left introduction rule tells us how it behaves in the premises of inferences, while the right introduction rule tells us how it behaves in the conclusions of inferences. Thus, the left introduction rule specifies the consequences of sentences containing the logical constant in question, while the right introduction rule specifies the grounds of such sentences," (1988, 55).

In this way, the meaning of a logical connective can be understood as given, in inferentialist terms, by the left and right rules in a sequent calclus. For instance, the meaning of conjunction is given by these rules here.

As Aurthur Prior (1960) famously showed, however, not just any set of left and right rules ought to count as defining a legitimate meaning. Consider the connective *tonk* who's meaning is purportedly given by the following rules:

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \textit{ tonk } B} \qquad\qquad \frac{\Gamma, B \vdash C}{\Gamma, A \textit{ tonk } B \vdash C}$$

So, if Γ implies A, then Γ implies A *tonk* B, and if Γ along with B implies some conclusion C, then Γ along with A *tonk* B implies C. These seem like perfectly coherent inferential rules. However, insofar as our consequence relation is *transitive*, so that we can link proofs together, introducing a connective into our logical language with these rules will trivialize the consequence relation of that langauge, making it the case that anything follows from anything else, as shown in the following proof of $p \vdash q$ for arbitrary p and q:

$$\frac{\dfrac{p \vdash p}{p \vdash p \textit{ tonk } q} \quad \dfrac{q \vdash q}{p \textit{ tonk } q \vdash q}}{p \vdash q}$$

Prior concludes with the challenge of specifying why the *and* rules count as defining a meaning, but the *tonk* rules don't. In an equally famous reply to Prior, Neul Belnap (1962) argues that a necessary condition for a set of rules counting as definitive of the meaning of a logical connective is that introducing a connective into a language with those rules constitute a *conservative extension* of that language, such that no new sequents containing only old vocabulary come to be derrivable as a result of introducing the new vocabulary.

In a natural deduction setting, Michael Dummett termed this requirement of conservativity as a requirement of *harmony* between the introduction and elimination rules: the elimination rules must not be too strong, relative to the introduction rules. In the proof-theoretic framework for natural deduction developed by Daq Prawitz, this requirement is demonstrated by a *reduction*, showing that, in any proof in which some compound formula is introcued only to be subsequently eliminated can be reduced to one that doesn't contain the detour through the introduction and elimination of that compound formula. One of the key insights of Kremer's paper is that, in a sequent calclus setting, the criterion of harmony can be understood as formally established by a proof of the eliminibility of the structural rule of *Cut*:[3]

$$\frac{\Gamma \vdash A \quad \Delta, A \vdash B}{\Gamma, \Delta \vdash B}\text{ Cut}$$

Here, in the single conclusion context, it's clear that Cut is a transitivity rule. It says that if A can be derived from some set of premises Γ, and, if A along with another set of premises Δ derives B, then B can be derived directly from Γ, Δ. In the proof of $p \vdash q$ by way of the *tonk* rules, shown above, the Cut rule needs to be used to derive the sequent $p \vdash q$. There's no way of derriving the same sequent without the use of Cut. In other words, in this proof, Cut is ineliminable. This contrasts the "improper" logical connective *tonk* with "proper" logical connectives like conjunction given again by the following rules:

[3]As a technical result, the equivalence of normalization in natural deduction and Cut elimination in the sequent calculus was noted already by Prawitz (and hinted at by Genzten), but Kremer's insight is to rearticul

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \mathrm{L}_\wedge \qquad\qquad \frac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \mathrm{R}_\wedge$$

For proofs involving these rules, Cut is eliminable. It's easy to see how, technically, the eliminability of Cut for sequent rules for some bit of logical vocabulary ensures that these rules are conservative, for the conclusions of those rules will always contain that vocabualry. The only rule that can take you from premise sequents containing that new vocabulary to a conclusion sequent containing only old vocabulary is Cut. So, if everything that one can with Cut can be derrived without Cut, this means that one cannot use the sequent rules for the new vocabulary to derrive sequents containing only old vocabulary.

Though it's clear that proving Cut Elimination amounts to proving conservativity, to appreciate how proving Cut Elimination amounts to proving a kind of harmony between the left and right rules, it is illuminating to actually look at the proof. The actual proof of Cut Elimination is an induction on Cut formula complexity in which we show, first, that Cut on atomic formulas is eliminable and second, that when we use Cut on a complex formula to derrive some sequent, we can always derrive the same sequent by using Cut on simpler formulas.[4] Here is the transformation showing that this inductive step holds for conjunction:

$$\frac{\dfrac{\Gamma \vdash A \quad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \mathrm{R}_\wedge \qquad \dfrac{\Delta, A, B \vdash C}{\Delta, A \wedge B \vdash C}\ \mathrm{L}_\wedge}{\Gamma, \Delta \vdash C}\ \mathrm{Cut}$$

$$\rightsquigarrow$$

$$\frac{\Gamma \vdash B \qquad \dfrac{\Gamma \vdash A \quad \Delta, A, B \vdash C}{\Gamma, \Delta, B \vdash C}\ \mathrm{Cut}}{\Gamma, \Delta \vdash C}\ \mathrm{Cut}$$

This shows that, by using Cut to link up a sequent in which $A \wedge B$ has been introduced on the right (by way of the right rules) with a sequent in which $A \wedge B$ has been introduced on the left (by way of the left rules), one cannot obtain anything that one could not already obtain by linking together the premises that that go into getting $A \wedge B$ on the right and $A \wedge B$ on the left.

[4]The induction on formula complexity involves a sub-induction

Thus, the rules definitive of the meaning of $A \wedge B$ are harmonious in that the *consequences* of $A \wedge B$ are not too strong relative to the *grounds* of $A \wedge B$. This is precisely what is not the case for *tonk*, as shown by the fact that no transformation of this form is possible. In this way, Michael puts forward a way of understanding the conceptual significance of the sequent calculus as providing an inferentialist account of meaning of the logical connectives, and the key theorem of the sequent calculus—Cut Elimination—as providing the key criterion of harmony required for a set of rules to count as actually defining a logical meaning.

2 A Brief Recent History of Brandom's Inferentialism

In a moment, I'm going to raise some problems with this conception of the sequent calculus, but, before I do, I just want to point out that Kremer was quite ahead of his time here in spelling out inferentialism in terms of the sequent calclus. It is only quite recently that Brandom has caught up and decided to do things in this manner. As a way of providing some further motivation for the use of the sequent calculus in the context of inferentialist semantics, let me now briefly overview where Brandom's inferentialism has gone in the thirty-five years since Kremer's paper.

When Kremer was at Pittsburgh in the mid 80s, drafts of *Making It Explicit* had been circulating at Pittsburgh for some time. The book, published in 1994, systematically lays out inferentialism as a global theory of meaning. However, it contains no formal framework for actually doing inferentialist semantics.[5] Brandom's first real attempt at such a formal framework didn't come until the formal incompatibility semantics put foward in his 2006 Locke Lectures (published in 2008 as *Between Saying and Doing*). This semantics, however, had a crucial problem: the consequence relations it defined were *montonic*. It was

[5]One step towards a formal inferentialist framework was made by Michael's brother Phillip, along with Mark Lance, who, the same as *Making It Explicit* was published, put forward a set of proof systems for conditionals meant to capture the notion of committive consequence that plays a central role in Brandom's work (Kremer and Lance 1994, 1996). These systems were quite limited in scope, however, with really just the conditional as the target connective, and didn't offer the prospect of a general framework for inferentialist semantics.

built into the semantic framework at ground level that if a set of sentences X is incompatible with a set of sentences Y, then any superset of X is incompatible with Y. The problem is that the concept of material incompatibility that the semantics is meant to be modeling simply doesn't work like this. For instance, "Sadie's a mammal" is incompatible with "Sadie lays eggs," but "Sadie's a mammal" along with "Sadie's a platypus" isn't incompatible with "Sadie lays eggs." Even just in the context of providing an account of the meanings of the logical connectives (bracketing the question of providing an account of the meanings of such expressions as "mammal" and "platypus"), this is a serious problem, since the sentences with which we use logical connectives include such sentences as these sentences about animals. The concept of negation defined in Brandom's incompatibility semantics, thus can't be the concept of negation that we're using when we infer from "Sadie's a mammal" to "Sadie doesn't lay eggs." Since it seems like we're deploying the very same concept of negation when we infer from "Sadie's a mammal" to "Sadie doesn't lay eggs" as when we infer from "The ball's red" to "The ball's not green," this is a serious problem.[6]

It is for this reason that, in the early 2010s, Brandom, moved by technical work by his student Ulf Hlobil, ended up coming around to formulating inferentialism in terms of the sequent calculus.[7] One feature of the sequent calculus is that it forces one to explicitly use structural rules such as Monotonicity, or, as Gentzen put it, *Weakening*:

$$\frac{\Gamma \vdash A}{\Gamma, B \vdash A} \text{ Weakening}$$

The fact that the use of a structural rule such as Weakening itself constitutes a logical step in the sequent calculus makes it possible construct *substructural logics*: logical systems that work without the use of such rules. Now, there are different reasons to want a logical system that works without such rules, but, for Brandom, the reason is so that the system is able to accomodate sequents for which they actually fail. Of course, Weakening holds for any strictly *logical*

[6]See Nickel (2013) for a criticism of this framework on these grounds.
[7]For Hlobil's statements of the view, see Hlobil

inference. If A logically entails B, then, no matter what premises you add to A, you'll still have a logical entailment. However, by rejecting Weakening, we can introduce into our logical system not just sequents encoding logical entailments, but sequents encoding defeasible material inferential relations as well. For instance, we can add, as an non-logical axiom of our sequent calculus, a sequent such as:

$$\textbf{bird} \vdash \textbf{flies}$$

and we can do this while maintaining

$$\textbf{bird}, \textbf{penguin} \nvdash \textbf{flies}$$

Now Gentzen's own sequent calculi require the structural rule of Weakening to function. Moreover, the connective rules enforce Weakening with conjuncts and disjuncts. For instance, Gentzen's left-conjunction rules are the following:

$$\frac{\Gamma, A \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \mathrm{L}_{\wedge_1} \qquad\qquad \frac{\Gamma, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \mathrm{L}_{\wedge_2}$$

These rules would let us reason from

$$\textbf{bird} \vdash \textbf{flies}$$

to

$$\textbf{bird} \ \wedge \ \textbf{penguin} \vdash \textbf{flies}.$$

And, of course, this is an unacceptable consequence. However, by tweaking the rules, we can avoid such consequences. In his 1944 dissertation, Oiva Ketonen put forward a classical sequent calclus in which not just Cut, but Weakening as well, is eliminable.[8] Here is Ketonen's classical sequent calculus:

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \mathrm{Ax}$$

Where Γ, Δ, and $\{A\}$ contain only atomics.

[8]And, moreover (and more technically significantly), Contraction is eliminable as well, but I'll ignore this fact here, as I am, for simplicity, treating sequents as relating sets.

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ L_{\neg} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ R_{\neg}$$

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L_{\wedge} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ R_{\wedge}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ L_{\vee} \qquad \frac{\Gamma \vdash A, B, \Delta}{\Gamma \vdash A \vee B, \Delta}\ R_{\vee}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma, B \vdash \Delta}{\Gamma, A \to B \vdash \Delta}\ L_{\to} \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \to B, \Delta}\ R_{\to}$$

Note that the axiom schema here is distinct from the more familiar axioms schema of *Reflexivity*: $A \vdash A$. Ketonen's axiom schema generalizes Reflexivity to allow for axioms in which additional formulas have been added in on the left or right. This builds in all of the Weakening one needs for classical logic in the axioms, and so Weakening as a structural rule can be eliminated. Because Weakening is eliminable, this system permits the addition of non-logical material axioms for which Weakening actually fails, and, unlike Gentzen's rules, the rules of this system play nicely with such axioms. For instance, if you look at the conjunction rules:

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ L_{\wedge} \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma \vdash B, \Delta}{\Gamma \vdash A \wedge B, \Delta}\ R_{\wedge}$$

you'll see that we can no longer derive

bird $\wedge$ **penguin** $\vdash$ **flies**.

from the sequent

bird $\vdash$ **flies**

We need the sequent

bird, **penguin** $\vdash$ **flies**

which we won't include as a material axiom, since it's not a good material inference.

The current formal inferentialist framework endorsed by Brandom involves simply adding to this sequent calculus additional non-logical material axioms for which Weakening may actually fail, such as the following:

1. **red** $\vdash$ **colored**
2. **red**, **green** $\vdash$
3. **bird** $\vdash$ **flies**
4. **mammal**, **lays eggs** $\vdash$

Weakening holds for the first two of these sequents, but it fails for the second two, and, notably, all of them are integrated into the same logical system whose rules are proposed as definitive of the meanings of the logical connectives. Thus, we are entitled to say that there is a single meaning of words like "not" and "and" which we grasp whether we're reasoning about monochromatic solids or animals. In this way, doing things in terms of the sequent calculus constitutes a definitive advance in the formal development of inferentialism.

This is the main way in which Brandom now formally conceives of the inferentialist program, and precisely the conception of the sequent calculus put forward by Kremer 35 years prior suggests itself here. This conception is explicit in the work of Brandom's recent student, Dan Kaplan, who's largely responsible for this current formal set-up. Kaplan introduces the sequent calculus formalization of inferentialism as follows:

> [T]he meaning of a sentence is given by its role as a premise and a conclusion in argument, or as I shall say: the meaning of p just is the contribution that p makes to the goodness of implication. Thus, we might understand the meaning of p as specified in:

$$\begin{array}{cc} \Gamma_1, p \vdash \Delta_1 & \Gamma_1 \vdash p, \Delta_1 \\ \Gamma_2, p \vdash \Delta_2 & \Gamma_2 \vdash p, \Delta_2 \\ \vdots & \vdots \\ \Gamma_n, p \vdash \Delta_n & \Gamma_n \vdash p, \Delta_n \\ \vdots & \vdots \end{array}$$

> On an inferentialist understanding of meaning, we treat the meaning of p as the contribution it makes to good inference above.

Of course, this is just the understanding that Kremer lays out. Brandom himself, however, is uneasy with this formulation. Let me explain why.

3 The Issue of Multiple Conclusions

You will note that the sequent calculus I've just shown features *multiple conclusions*. This is an essential feature of this system, and it is an essential feature of classical sequent calclui in general. Now, a multiple conclusion sequent is not a collection of single conclusion sequents. That is, the sequent $\Gamma \vdash \Delta$ is not to be understood as shorthand for the set of sequents $\Gamma \vdash A$ for each $A \in \Delta$. That would be to take the elements of Δ to be collected *conjunctively*, and the crucial idea of a multiple conclusion sequent calculus is that, whereas the *premises* of a sequent are collected conjunctively, the *conclusions* are collected *disjunctively*. This raises a serious interpretive problem. In a single conclusion sequent, we can read what goes to the left of the turnstile as the premises of an argument, and what goes to the right of the turnstile as the conclusion. We have a clear pre-theoretical grip on what it is for a set of premises, taken together, to imply a conclusion. That is, we have a clear pre-theoretical grip of what it is for an argument of the form "A, B, C, therefore D" to be a good one. We employ such arguments in our everyday practices of reasoning, and we have a grip on the pragmatic force such arguments are supposed to have. We also have a fine enough pre-theoretical grip on an argument of the form "A, B, C, therefore D, E," where the conclusions are taken conjunctively. But the idea of an "argument" with "premises" A, B, C and the "conclusions" D, E, where the "conclusions" D and E are to be taken *disjunctively* does not find much pre-theoretical traction in our actual practices of reasoning. Of course, we have a grip of what it is for a set of premises to entail single conclusion that is a *disjunction*, but the multiple conclusions of a sequent are not to be interpreted in that way any more than the multiple premises are to be interpreted as a single conjunction, as doing so would preclude us from being able to appeal to such sequents in giving an account of the meanings of these propositional connectives.

The issue of making good sense of multiple conclusion sequents—and doing so in a way that does not pressupose grasp of the logical connectives whose meanings they are supposed to formally accounting for—has long been the bugbear haunting proponents of multiple conclusion sequent calculi in the context of inferentialist semantics. Many proponents of inferentialist semantics remain convinced that, as Florian Steinberger (2011) puts it, "Conclusions Should Remain Single." Given this issue, though Kremer's explication of the conceptual significance of the sequent calculus surely holds up just fine for the *intuitionistic* sequent calculus, which features only single conclusion sequents, it is doubtful that the case of the *classical* sequent calculus can be understood so straightforwardly. To illustrate the problem here, consider the negation rules of Ketonen's sequent calclus (which are the same as those in Gentzen's LK):

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \mathrm{L}_\neg \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \mathrm{R}_\neg$$

To a large extent, it is these rules that elevate the proof-theoretic status of the classical sequent-calclus above that of natural deduction for classical logic. For, while the rules for negation in natural deduction rules are not harmonious (the double negation elimination rule is too strong relative to the introduction rule of reductio ad absurdum)—these rules clearly are harmonious by the sequent calculus's criterion of harmony: Cut Elimination. However, despite the proof-theoretic failing of the negation rules for natural deduction, those rules *at least make intuitive sense*, and the same cannot be said for these negation rules. Indeed, it's not at all clear what these rules *actually say*.

On Kremer's proposed reading of the sequent calclus, these rules for negation respectively tell us that the inferential role of *A as a conclusion* is the same as the inferential role of ¬*A as a premise*, and the inferential role of *A as a premise* is the same as the inferential role of ¬*A as a conclusion*. Clearly, this does not make any sense apart from thinking of conclusions as *multiple*, and we simply don't have a clear pre-theoretical sense of multiple conclusion "implications" to appeal to here. Of course, we *can* make sense of them if we appeal to our understanding of disjunction and its interaction with negation. For instance, considering just the case where Γ and Δ are single formulas, we can rewrite the negation rules as follows:

$$\frac{B \vdash A \vee C}{B, \neg A \vdash C} \qquad\qquad \frac{B, A \vdash C}{B \vdash \neg A \vee C}$$

Clearly, if B implies A or C, then B along with $\neg A$ must imply C, and if B along with A implies C, then B must imply either $\neg A$ or C (after all, implying both A and $\neg C$ would contradict the premise). By appealing to our understanding of disjunction and its interaction with negation, we can see intuitively that these rules are indeed sound. But, of course, we cannot appeal to this understanding if we want to appeal to these rules in order to account of the sense of these logical constants. Alternately, we can make sense of the soundness of the rules by appealing to a semantic interpretation of sequents according to which a sequent is valid just in case there's no valuation such that all of the premises are true and all of the conclusions are false, but, once again, such an appeal is ruled out insofar as we're aiming to provide a proof-theoretic rather than model-theoretic account of the meanings of the logical connectives. The same problem can be raised for the other connective rules.

This is not a new problem, and I'm not going to give an exhaustive account of proposed solutions and their potential issues here.[9] Instead, I want to consider one prominent response to this problem that has emerged in recent years that has transformed many people's conception of the basic topic of logic. I will raise a problem for this response as well, but I think it takes us in the right direction.

4 Restall's Bilateralism

In response to these concerns about multiple conclusions, Greg Restall (2005, 2009) has proposed a reading of multiple conclusion sequents according to which the turnstile plays the role not of separating *premises* from *conclusions*

[9] See Steinberger (2011) for such a discussion.

but of separating *affirmations* from *denials*.[10] On this *bilateralist* reading of multiple conclusion sequents, a sequent of the form $\Gamma \vdash \Delta$ says that the position consisting in affirming everything in Γ and denying everything in Δ is incoherent or "out of bounds." To see how this interpretation resolves our problem with multiple conclusions, consider again the negation rules of the classical sequent calculus:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \mathrm{L}_{\neg} \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \mathrm{R}_{\neg}$$

On the bilateralist interpretation, the left rule says that if the position consisting in affirming everything in Γ, denying everything in Δ, and denying A is out of bounds, then the position consisting in affirming everything in Γ, denying everything in Δ, and affirming $\neg A$ is out of bounds. The right rule says that if the position consisting in affirming everything in Γ, denying everything in Δ, and affirming A is out of bounds, then the position consisting in affirming everything in Γ, denying everything in Δ, and denying $\neg A$ is out of bounds. So, understanding the significance of speech acts in terms of their contribution to the (in)coherence of positions, this rule tells us that an affirmation of $\neg A$ has the same significance as a denial of A, and a denial of $\neg A$ has the same significance as an affirmation of A. By substituting talk of premises and conclusions with talk of affirmations and denials, we get a conception of the sequent rules for negation that makes clear intuitive sense.

There are two points I want to make about this bilateralist reading of the multiple conclusion sequent calculus that will eventually bring us to the conception of the sequent calculus I want to endorse. The first point is that, if we do read the multiple conclusion sequent calculus in this bilateralist fashion, the notation of writing sequents as formulas of the form $\Gamma \vdash \Delta$ because a bit

[10]Or assertions from denials, or acceptances from rejections, or some other pair of opposite linguistic or mental acts. I speak here in terms of affirmations and denials, following Rumfitt (2000), but the particular pair of opposite acts one opts for doesn't matter for our purposes. Elsewhere (Simonelli 2022), I have articualted bilateralism in terms of the opposite normative statuses of *commitment* and *preclusion of entitlement*. That reading is also available here, with minor tweaking of the sense of the turnstile. Alternately, following Hlobil (2021), one might think of this bilateralist set up in terms of *truth* and *falsity*. What follows can be translated into any of these vocabularies.

misleading, since $\Gamma \vdash \Delta$ does not really express a relation of *implication* or *consequence* betweent he sets of sentences Γ and Δ at all. Rather, it expresses the *incoherence* of the set consisting in the affirmation of all sentences in Γ and the denial of all of the sentences in Δ. A more perspicuous notation, then, would be to follow bilateralists such as Smiley (1996) and Rumfit (2000) and formulate the bilateralism of Restall and Ripley in a bilateral *notation* in which formula are positively or negatively *signed*, explicitly marking affirmations and denials in the notation itself. In such a system, in order to constitute a well-formed formula, a sentence must be prefaced with a positive or negative force-marker, expressing either the affirmation or denial of that sentence. Thus, the affirmation of a sentence A can be written as $+\langle A\rangle$, and the denial for A can be written as $-\langle A\rangle$. Unlike a negation operator, these force-markers are neither embedable or iterable; there must always be exactly one force-marker and it must always be prefixed to a whole sentence. So, for instance, although both $+\langle p \wedge \neg q\rangle$ and $-\langle \neg p\rangle$ are well-formed, neither $+\langle p \wedge -\langle q\rangle\rangle$ nor $-\langle -\langle p\rangle\rangle$ are well-formed.

Having introduced this notation, it is simple to translate multiple conclusion sequents, interpreted in bilateralist fashion, to sets of positively and negatively signed formulas: a sequent of the form $\Gamma \vdash \Delta$, where Γ and Δ contained unsigned sentences, is mapped to a set of signed sentences Θ, where $\Theta = \{+\langle A\rangle \mid A \in \Gamma\} \cup \{-\langle A\rangle \mid A \in \Delta\}$. That is, Θ is set the consisting in a formula of the form $+\langle A\rangle$ for sentence A in Γ along with a formula of the form $-\langle A\rangle$ for every sentence A in Δ. (In what follows, I maintain the convention of using Γ and Δ for sets of unsigned sentences and Θ and Λ for sets of signed sentences.) In this explicitly bilateral notation, rather proof rules for inferring between sequents consisting in sets of sentences flaking a turnstile on each side, we have rules for inferring between sets of signed formulas. A set's being derivable means that the position consisting in all of the affirmations and denials in that set is out of bounds, and a sentence is a logical theorem just in case the singleton containing the denial of that sentence is derrivable from logical axioms.

To formulate the previously considered sequent calculus in this new notation, let us introduce the further notational convention of using lower-case Greek letters to indicate signed formulas, which may be either affirmations or denials, where starring a signed formula yields the oppositely signed formula. So, if φ is the affirmation of A, then φ^* is the denial of A and vice versa. With these notational conventions, Ketonen's sequent calculus, on Restall's bilateral understanding of it, comes out, in explicitly bilateral notation, as follows:

$$\frac{}{\Theta, \varphi, \varphi^*}\ \text{Ax}$$

Where Θ and $\{\varphi\}$ contain only signed atomics.

$$\frac{\Theta, -\langle A\rangle}{\Theta, +\langle \neg A\rangle}\ +_\neg \qquad \frac{\Theta, +\langle A\rangle}{\Theta, -\langle \neg A\rangle}\ -_\neg$$

$$\frac{\Theta, +\langle A\rangle, +\langle B\rangle}{\Theta, +\langle A \wedge B\rangle}\ +_\wedge \qquad \frac{\Theta, -\langle A\rangle \quad \Theta, -\langle B\rangle}{\Theta, -\langle A \wedge B\rangle}\ -_\wedge$$

$$\frac{\Theta, +\langle A\rangle \quad \Theta, +\langle B\rangle}{\Theta, +\langle A \vee B\rangle}\ +_\vee \qquad \frac{\Theta, -\langle A\rangle, -\langle B\rangle}{\Theta, -\langle A \vee B\rangle}\ -_\vee$$

$$\frac{\Theta, -\langle A\rangle \quad \Theta, +\langle B\rangle}{\Theta, +\langle A \to B\rangle}\ +_\to \qquad \frac{\Theta, +\langle A\rangle, -\langle B\rangle}{\Theta, -\langle A \to B\rangle}\ -_\to$$

In this notation, it's explicit that what the axiom schema says is that any position that consists in affirming some atomic sentence and also denying that atomic sentence is incoherent. Moreover, if you look at the negation rules here, you can see that they explicitly *show* in the bilateral notation itself, what the more familiar formulation of them *says*, on a bilateralist interpretation of them.

Now, let us turn to consider the significance of Cut. In the multiple conclusion setting the rule of Cut is the following:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'}\ \text{Cut}$$

In this bilateral notation, we can put the Cut rule more simply as follows, where, once again Θ and Λ are sets of signed formulas:

$$\frac{\Theta, \varphi \quad \Lambda, \varphi^*}{\Theta, \Lambda}\ \text{Cut}$$

Cut is now no longer straightforwardly a transitivity rule, as it is in the single conclusion case. The intuitive reasoning behind this structural rule is as follows. If the set Θ, φ is out of bounds, then, if one takes all of the stances in Θ, then one cannot take stance φ.[11] If the set Λ, φ^* is out of bounds, then, if one takes all of the stances in Λ, then one cannot coherently take stance φ^*. So, if both Θ, φ and Λ, φ^* are out of bounds, then, if one takes all of the stances in Θ, Λ, then one cannot coherently take either φ or φ^*. Given the rule of Cut, this means that Θ, Λ must itself be incoherent. As articulated by Restall, this amounts to an *extensibility condition* on coherent position: any coherent position can't rule out both positive and negative stances towards some sentence; it must be coherently extendable to one of these stances. This is, of course, indeed the case for the classical sequent calculus, as is shown by a proof that the rule of Cut is eliminable. In this context, the sort of harmony established by the proof of Cut Elimination is no longer a harmony between *left* and *right* rules, but, rather, between *positive* and *negative* rules. It thus encodes the fact that the conditions ruling out the *affirmation* of a logically compound sentence are not too strong or weak relative to the conditions for ruling out the *denial* of that sentence.

Restall's bilateralism does resolve the problem of making sense of the multiple conclusion connective rules. However, it seems to come at a price: logic no longer concerns consequnce at all. It only concerns (in)coherence. The legitimacy of this move from from thinking of the rules governing the use of logical expressions in terms of *consequence* to thinking about them in terms of constraints on *coherence* is given some expression by Wilfrid Sellars. In "Meaning as Functional Classification," Sellars writes:

> It should be stressed that the uniformities involved in meaningful verbal behavior include *negative* uniformities, i.e., the avoidance of certain combinations, as well as *positive* uniformities, i.e., uniformities of concomitance. Indeed negative uniformities play by far the

[11] I use the term "stance" here and in what follows to speak neutrally about affirmations and denials. I use this term ambiguously (just as "affirmation" and "denial" are used ambiguously in natural language) to speak both of affirmations and denials towards particular propositions (notated with expressions like φ and φ^*) and affirmations and denials in abstraction from attachment to any particular propositions (notated with expressions like like $\boldsymbol{a}$ and $\boldsymbol{a}^*$).

> more important role, and the rules which govern them are to be construed as *constraints* rather than incentives, (Sellars 1974, 86).

There is surely some truth to Sellars's statement here. Still, it feels like this bilateralist understanding of logic, according to which logic's concern is *just* incoherence, is incomplete. Surely logic doesn't just make *negative* demands on what one *can't* maintain; it also makes *positive* demands on what one *must* maintain, at lest insofar as one maintains or is committed to maintaining other things. It is this positive notion of committive consequence, I take it, that Ian Rumfit (2008) is speaking of when he speaks of "the force of consequence" which he criticizes a Restall's understanding of the turnstile for lacking.[12] In illustrating this notion of force, he considers a hypothetical (or perhaps actual) exchange with a student in one of his seminars:

> What do you mean, you refuse to accept B? You continue to adhere to A, and I've shown you that B follows from A.

Rumfit takes it that, given the student's acceptance of *A* and acknowledgment of the fact that *B* follows from *A*, she's *obliged* to accept *B*. On Restall's understanding of validity, all one can say here is that she is *precluded from being entitled* to deny *B*. Of course, Rumfitt acknowledges that is indeed the case, but he thinks is crucial that we be able to say some further here as well: that, given the stances that she has taken, she's *committed* to affirming *B*.

The core problem with Restall's bilateralist interpretation of the sequent calculus is that the very notion of something's "following from" something else drops out of the understanding of the logical system entirely. If the formal framework in which we pursue an inferentialist semantics for the logical connectives is a multiple conclusion sequent calculus, and we understand this sequent calculus in Restall's fashion, where there is no notion of an inferential relation at all, calling our framework "inferentialist" would be quite an irony, to put it mildly. It is natural to wonder whether we can have our cake and eat it too: whether we can have the intuitive understanding the connective rules of the multiple conclusion sequent calculus that the bilateralist proposes,

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while nevertheless maintaining that our logical system doesn't just concern coherence, but consequence as well. I'll now propose a new bilateral system that enables us to do just that.

5 A Broader Bilateralist Logic

To introduce the bilateral sequent calculus I'm going to propose, let me first return to the unilateral sequent calculus to explain a feature that I glossed over earlier. Earlier, when I was discussing the sequent calculus formalization of inferentialism, I showed you the following two sequents:

$$\textbf{red} \vdash \textbf{colored}$$
$$\textbf{red}, \textbf{green} \vdash$$

The significance of the first sequent was explicitly explianed, but you might have been wondering the meaning of the second of these two sequents, which has two sentences on the left side and an empty right side. Given the negation rules we discussed earlier:

$$\frac{\Gamma \vdash A, \Delta}{\Gamma, \neg A \vdash \Delta}\ \mathrm{L}_\neg \qquad\qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \neg A, \Delta}\ \mathrm{R}_\neg$$

such a sequent can be understood as encoding the *incompatibility* between these sentences, since, from these rules and this sequent, we can derrive:

$$\textbf{red} \vdash \neg\textbf{green}$$

and

$$\textbf{green} \vdash \neg\textbf{red}$$

In general, a sequent of the form $\Gamma \vdash$ can be understood as formally encoding the fact that the set of sentences Γ is incoherent, as borne out by the fact that, for all Γ', where $\Gamma' = \Gamma \setminus \{A\}$ with $A \in \Gamma$, $\Gamma' \vdash \neg A$.

In a bilateral system, we can get the same behavior at the structural level by way of following pair of rules:

$$\frac{\Theta, \varphi \vdash}{\Theta \vdash \varphi^*}\ \text{Out} \qquad\qquad \frac{\Theta \vdash \varphi}{\Theta, \varphi^* \vdash}\ \text{In}$$

The Out rule can be understood as saying that, if the position consisting in all of the stances in Θ along with stance φ is incoherent, then Θ commits one to taking the opposite stance φ^*, whereas the In rule can be understood as saying that, if Θ commits one to taking the stance φ, then the position consisting Θ along with the opposite stance φ^* is incoherent. With these rules in view, consider the following sequent:

$$+\langle \textbf{red} \rangle, +\langle \textbf{green} \rangle \vdash$$

This says that the position consisting in affirming "x is red" and affirming "x is green" is incoherent. The incoherence of the position consisting in both of these affirmations can be understood in terms of the fact that affirming "x is red" commits one to denying "x is green" and affirming "x is green" commits one to denying "x is red," where the relation between all of these inocherence and incompatibility facts is codified by In and Out, as, given these rules, this sequent is equivalent to this one:

$$+\langle \textbf{red} \rangle \vdash -\langle \textbf{green} \rangle$$

and this one:

$$+\langle \textbf{green} \rangle \vdash -\langle \textbf{red} \rangle$$

Now, recall, in the previous calculus, we just had sets of signed formulas and no turnstile, and a set's being derivable means that the position consisting in all of the stances in that set is incoherent. We can now make this fact explicit in the logical system: we simply add a turnstile to the right of all of the rules:

$$\frac{}{\Theta, \varphi, \varphi^* \vdash}\ \text{Ax}$$

Where Θ and $\{\varphi\}$ contain only signed atomics.

$$\frac{\Theta, -\langle A \rangle \vdash}{\Theta, +\langle \neg A \rangle \vdash}\ +_\neg \qquad\qquad \frac{\Theta, +\langle A \rangle \vdash}{\Theta, -\langle \neg A \rangle \vdash}\ -_\neg$$

$$\frac{\Theta, +\langle A\rangle, +\langle B\rangle \vdash}{\Theta, +\langle A \wedge B\rangle \vdash} +_\wedge \qquad \frac{\Theta, -\langle A\rangle \vdash \quad \Theta, -\langle B\rangle \vdash}{\Theta, -\langle A \wedge B\rangle \vdash} -_\wedge$$

$$\frac{\Theta, +\langle A\rangle \vdash \quad \Theta, +\langle B\rangle \vdash}{\Theta, +\langle A \vee B\rangle \vdash} +_\vee \qquad \frac{\Theta, -\langle A\rangle, -\langle B\rangle \vdash}{\Theta, -\langle A \vee B\rangle \vdash} -_\vee$$

$$\frac{\Theta, -\langle A\rangle \vdash \quad \Theta, +\langle B\rangle \vdash}{\Theta, +\langle A \rightarrow B\rangle \vdash} +_\rightarrow \qquad \frac{\Theta, +\langle A\rangle, -\langle B\rangle \vdash}{\Theta, -\langle A \rightarrow B\rangle \vdash} -_\rightarrow$$

To turn this calculus of incoherence into a calculus of consequence, we now simply rewrite all of these sequents so that bottom sequents all have consequents in the way that having In and Out in our system entitles us to do. There are different ways to do this, but I propose the following sequent calculus as providing the most intuitive inferential articulation of the meanings of the classical connectives:[13]

$$\frac{}{\Theta, \varphi \vdash \varphi} \text{Ax} \qquad \frac{\Theta \vdash \varphi}{\Theta, \varphi^* \vdash} \text{In} \qquad \frac{\Theta, \varphi \vdash}{\Theta \vdash \varphi^*} \text{Out}$$

Where Θ and $\{\varphi\}$ contain only signed atomics.

$$\frac{\Theta \vdash -\langle A\rangle}{\Theta \vdash +\langle \neg A\rangle} +_\neg \qquad \frac{\Theta \vdash +\langle A\rangle}{\Theta \vdash -\langle \neg A\rangle} -_\neg$$

$$\frac{\Theta \vdash +\langle A\rangle \quad \Theta \vdash +\langle B\rangle}{\Theta \vdash +\langle A \wedge B\rangle} +_\wedge \qquad \frac{\Theta, +\langle A\rangle, +\langle B\rangle \vdash}{\Theta \vdash -\langle A \wedge B\rangle} -_\wedge$$

$$\frac{\Theta - \langle A\rangle, -\langle B\rangle \vdash}{\Theta \vdash +\langle A \vee B\rangle} +_\vee \qquad \frac{\Theta \vdash -\langle A\rangle \quad \Theta \vdash -\langle B\rangle}{\Theta \vdash -\langle A \vee B\rangle} -_\vee$$

$$\frac{\Theta, +\langle A\rangle, -\langle B\rangle \vdash}{\Theta \vdash +\langle A \rightarrow B\rangle} +_\rightarrow \qquad \frac{\Theta \vdash +\langle A\rangle \quad \Theta \vdash -\langle B\rangle}{\Theta \vdash -\langle A \rightarrow B\rangle} -_\rightarrow$$

[13]See Simonelli (forthcoming) for an extended defense of this sequent calculus in the context of proof-theoretic semantics. As you will see, all of the binary connective rules have the same basic form. Where $\boldsymbol{a}$ and $\boldsymbol{b}$ are variables over signs, and * is a function that maps + to − and − to +, the rules for all of the binary connectives of classical logic are given by the following rule schema:

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle} \boldsymbol{c}_\circ \qquad \frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle} \boldsymbol{c}^*{}_\circ$$

Clearly, given In and Out, this system is equivalent to the previous one; both are equivalent to Ketonen's multiple conclusion classical sequent calculus. However, only this latter system can properly be conceived of as providing *introduction rules* for positively and negatively signed formulas, specifying the grounds for affirming or denying a logically complex sentence.[14] In particular, the negation rules are just the introduction rules proposed by Rumfitt (2000) in the context of his natural deduction system. They say that if a position Θ commits one to denying A, then Θ commits one to affirming $\neg A$, and if Θ commits one to affirming A, then Θ commits one to denying $\neg A$. The binary connective rules say when a set of positions Θ commits one to a positive or negative stance towards a sentence containing some binary connective.

To explain the conceptual significance of the binary connective rules, and, in particular, the negative conjunction and positive disjunction rules, which likely look less familiar than the others, note first that In and Out jointly yield the the structural rule that Smiley (1996, 5) calls "Reversal":

$$\frac{\Gamma, A \vdash B}{\Gamma, B^* \vdash A^*}\ \text{Reversal}$$

Indeed, the pair of rules, In an Out, might alternately by formulated as simply Reversal where $\{A\}$ or $\{B\}$ can be null. With this structural rule in view, consider the negative conjunction rule:

$$\frac{\Theta, +\langle A\rangle \vdash -\langle B\rangle}{\Theta \vdash -\langle A \wedge B\rangle}\ -_{\wedge}$$

This rule says that if a set of stances Θ along with an affirmation of A commits one to denying B, then Θ commits one to denying $A \wedge B$. Note that, given Reversal, if Θ along with an affirmation of A commits one to denying A, then, just as well, Θ along with an affirmation of B commits one to denying A. So, essentially, this rule for conjunction says that you're committed to denying a conjunction just in case affirming one of the conjuncts commits you to denying the other. Dually, the positive disjunction rule:

[14]See Steinberger (2011, 349-353) for a criticism of Restall's bilateralism on account of the fact that it fails to provide such rules.

$$\frac{\Theta, -\langle A\rangle \vdash +\langle B\rangle}{\Theta \vdash +\langle A \vee B\rangle} +_{\vee}$$

says that you're committed to affirming a disjunction just in case denying one of the disjuncts commits you to affirming the other. In this way, the premise of the negative conjunction rule directly encodes a relation of *incompatibility* or *contraiety* obtaining between the conjuncts, relative to one's set of stances, whereas and the positive disjunction directly encodes a relation of *subcontraiety* obtaining between the disjuncts, relative to one's set of stances.

I claim that this bilateral sequent system vindicates the sequent calculus approach to inferentialism for classical logic: one can reasonably maintain that the rules of this system are definitive of the meanings of the classical connectives. To officially show that these rules qualify for defining meanings, let us now consider the harmony between the positive and negative rules of this system, formally established by the proof of Cut Elimination. In this system, the rule of Cut can be formulated as follows:

$$\frac{\Theta \vdash \varphi \quad \Lambda \vdash \varphi^{*}}{\Theta, \Lambda \vdash} \text{Cut}$$

This says that if the position Θ commits one to the stance φ, and the position Λ commits one to the opposite stance φ^{*}, then the position Θ, Λ is incoherent. Formulated as such, Cut is not a kind *transitivity* principle, but, rather, a kind of *reductio* principle, enabling one to include the incoherence of a position from the fact that it commits one to opposite stances towards some sentence. The crucial case in the proof of Cut Elimination for these connective rules shows that the positive and negative introduction rules are such that the positions required to commit one to opposite stances towards a logically compound sentence are themselves incompatible. For instance, in the case of conjunction, if a position Θ, Λ is incoherent in that Θ commits one to affirming $A \wedge B$ and Λ commits one to denying $A \wedge B$, then, even without the introduction of the opposite stances towards conjunction, Θ, Λ is *already* incoherent in that it commits one to opposite stances towards the conjuncts. Officially, we show this with the following transformation:

$$\dfrac{\dfrac{\Theta \vdash +\langle A\rangle \quad \Theta \vdash +\langle B\rangle}{\Theta \vdash +\langle A \wedge B\rangle}{+_\wedge} \qquad \dfrac{\Lambda, +\langle A\rangle \vdash -\langle B\rangle}{\Lambda \vdash -\langle A \wedge B\rangle}{-_\wedge}}{\Theta, \Lambda \vdash}\,\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\Theta \vdash +\langle A\rangle \qquad \dfrac{\dfrac{\Theta \vdash +\langle B\rangle \quad \Lambda, +\langle A\rangle \vdash -\langle B\rangle}{\Theta, \Lambda, +\langle A\rangle \vdash}\,\text{Cut}}{\Theta, \Lambda \vdash -\langle A\rangle}\,\text{Out}}{\Theta, \Lambda \vdash}\,\text{Cut}$$

Spelling out what is shown here, in the first proof we show what has to be the case about Θ and Λ for these positions to respectively commit one to affirming and denying $A \wedge B$. Given that Θ commits one to affirming $A \wedge B$, Θ commits one to affirming A and Θ commits one to affirming B. Given that Λ commits one to denying $A \wedge B$, Λ along with an affirmation of A commits one to denying B. In the second proof, we show that, given these facts about Θ and Λ, we can conclude that the position Θ, Λ is incoherent even apart from the introduction of opposite stances towards $A \wedge B$. Since Θ commits one to affirming B and Λ along with an affirmation of A commits to denying B, Θ along with Λ along with an affirmation A is incoherent, and that means that Θ along with Λ commits one to denying A. But Θ also commits one to affirming A. So Θ, Λ is incoherent.

In the actual proof of Cut Elimination proof, we show first that Cut on atomic formulas in eliminable, and we then induct on formula complexity to show that Cut on logically complex formulas can always be reduced to Cut on simpler formulas, so Cut in general is eliminable.[15] Transformations like this establish that inductive step. Now, the proof in the solely left-sided system shown earlier, which is a direct translation of the proof in the standard unilateral sequent notation, is a bit simpler, since one doesn't need to deal with the complication of using In and Out as one does here. However, I submit that the proof of Cut Elimination in this system—in which transformations like this figure—makes mainfest in the notation itself exactly how we should understand the conceptual significance of the proof of Cut Elimination for the classical sequent calculus. As Kremer says, the proof of Cut Elimination establishes harmony among the rules. However, the harmony established

[15]See Simonelli (2024) for a generalized Cut Elimination proof in this sort of bilateral system.

by Cut Elimination is not between the rules for inferring *to* a sentence as a conclusion and rules for inferring *from* that sentence as a premise; rather, Cut Elimination establishes harmony between the rules for *affirming* a sentence and rules for *denying* that sentence.

What, however, of the original understanding of Cut according to which it expresses a principle of Transitivity, and the understanding Cut Elimination as demonstrating harmony between the grounds for asserting a logical formula and consequences of asserting it? Is that understanding lost in this sort of sequent system? Not in the least. Unlike the previous bilateral system, we are able to recover all of this thinking about consequence in this bilateral system. Note first that, while I have provided just right rules, codifying the *grounds* for affirming or denying a sentence, corresponding left rules, codifying the *consequences* of affirming or denying a sentence, can be obtained immediately via Reversal. For instance, we have the following derrived left rules for conjunction:

$$\frac{\Theta, +\langle A\rangle, +\langle B\rangle \vdash \varphi}{\Theta, +\langle A \wedge B\rangle \vdash \varphi}\ L_{+_\wedge} \qquad\qquad \frac{\Theta, -\langle A\rangle \vdash \varphi \quad \Theta, -\langle B\rangle \vdash \varphi}{\Theta, -\langle A \wedge B\rangle \vdash \varphi}\ L_{-_\wedge}$$

Now, consider the following rule, which I'll just call "Transitivity":

$$\frac{\Theta \vdash \varphi \quad \Lambda, \varphi \vdash \psi}{\Theta, \Lambda \vdash \psi}\ \text{Transitivity}$$

This just is the single conclusion Cut rule shown earlier, but with signed formulas rather than unsigned formulas. Adapting Kremer's explication of the significance of the eliminability of this rule to this bilateral context, proving the eliminability of this rule establishes that the consequences of affirming or denying a logically compound sentence are not too strong relative to the grounds of affirming or denying that sentence. In this system, the eliminability of Transitivity follows directly from the eliminability of Cut, as Transitivity can be directly derived from Cut as follows:

$$\dfrac{\dfrac{\Theta \vdash \varphi \quad \dfrac{\Lambda, \varphi \vdash \psi}{\Lambda, \psi^* \vdash \varphi^*}\ \text{Reversal}}{\Theta, \Lambda, \psi^* \vdash}\ \text{Cut}}{\Theta, \Lambda \vdash \psi}\ \text{Out}$$

So, when it comes to coherence and consequence, one can truly have one's cake and eat it too.

6 Conclusion

My aim here has been to follow Michael Kremer in trying to understand the philosophical significance of the sequent calculus when it comes to giving an account of the meanings of the logical connectives. One thing I hope to have shown is that the *interpretive* task, of trying to make sense of such systems as Ketonen's classical sequent calculus, is often not seperable from the *logical* task, of developing new logical notations and systems involving their use. There is, I take it, a Wittgensteinian moral here in the philosophy of logic. The core concepts we deploy to *make sense* of logical systems, such as affirmation and denial or truth and falsity, are themselves conferred by our use of linguistic expressions, and formally codifying this use *in* a logical system can help bring to reflective consciousness the core concepts that are articulative of our capacity for logical thought. In his book *The Logical Alien*, James Conant classifies someone engaged in distinctively *philosophical* logic as aiming "to achieve a self-understanding of what she, the logical subject, is doing when she thinks," (358). I hope I've done a bit to show how work in *formal* logic, of the sort Kremer pursued early in his career and I have pursueed here, might be understood as oriented towards the acheivment of that aim.

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