$\mathbf{RWB}^{-}_{\to,\neg,\circ}$ Does Not Satisfy Variable Sharing

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Abstract

Tedder recently introduced a connexive extension of the contraction-free relevant logic $\mathbf{RW}^{-}$, called $\mathbf{RWB}^{-}_{\to,\neg,\circ}$, and left open whether it satisfies the variable-sharing property. We answer the question negatively. Providing two examples of the failure, we show

$$\vdash_{\mathbf{RWB}^{-}_{\to,\neg,\circ}} (\neg q \to q) \to (p \to \neg p).$$

and

$$\vdash_{\mathbf{RWB}^{-}_{\to,\neg,\circ}} (p \to p) \to (q \to q).$$

The derivations do not use fusion, so the failure already occurs in $\mathbf{RWB}^{-}_{\to,\neg}$.

In [1], Tedder considers the Boethius extension of the intensional fragment of $\mathbf{RW}^{-}$. Its language contains implication $\to$ and negation $\neg$. The axioms are the following:

$$A \to A, \tag{Id}$$
$$\neg\neg A \to A, \tag{DNe}$$
$$A \to \neg\neg A, \tag{DNi}$$
$$(A \to B) \to ((C \to A) \to (C \to B)), \tag{B}$$
$$(A \to B) \to ((B \to C) \to (A \to C)), \tag{B'}$$
$$(A \to (B \to C)) \to (B \to (A \to C)), \tag{C}$$
$$(A \to B) \to \neg(A \to \neg B), \tag{Boe_1}$$
$$(A \to \neg B) \to \neg(A \to B). \tag{Boe_2}$$

The system is closed under substitution and modus ponens, as well as the following rules:[1]

$$\frac{A \to B}{\neg B \to \neg A}\ (r\mathrm{Cont}), \qquad \frac{A \to B}{(C \to A) \to (C \to B)}\ (rB), \qquad \frac{A \to B}{(B \to C) \to (A \to C)}\ (rB').$$

The fusion expansion $\mathbf{RWB}^{-}_{\to,\neg,\circ}$ adds $\circ$ by the reversible residuation rule

$$\frac{(A \circ B) \to C}{A \to (B \to C)} \quad \text{and} \quad \frac{A \to (B \to C)}{(A \circ B) \to C}.$$

This rule will not be used in the result that follows, which answers the following question:

[1] Following Tedder's definition of $\mathbf{B}$ in §2.2, we use rCont for the rule displayed here. A bit confusingly, Tedder also uses the label rCont for the rule $\frac{A \to \neg B}{B \to \neg A}$ in his presentation of the system he calls "Thing$_3$." In the presence of (DNi) and rHS, the two forms are interderivable, and so nothing in what follows hangs on this.

Open Problem 1 (Tedder's Problem 4.7)**.** *Does* $\mathbf{RWB}^{-}_{\to,\neg,\circ}$ *satisfy variable sharing? That is, does*

$$\vdash_{\mathbf{RWB}^{-}_{\to,\neg,\circ}} A \to B \quad \Longrightarrow \quad \mathrm{Var}(A) \cap \mathrm{Var}(B) \neq \varnothing$$

hold for all formulas A *and* B*?*

Theorem 1. *If* p *and* q *are distinct propositional variables, then*

$$\vdash_{\mathbf{RWB}^{-}_{\to,\neg}} (\neg q \to q) \to (p \to \neg p).$$

Consequently, neither $\mathbf{RWB}^{-}_{\to,\neg}$ *nor* $\mathbf{RWB}^{-}_{\to,\neg,\circ}$ *satisfies variable sharing.*

Proof. Write $\vdash$ for theoremhood in $\mathbf{RWB}^{-}_{\to,\neg}$. In addition to the displayed rules, we freely use hypothetical syllogism and the exchange rule

$$\frac{A \to B \qquad B \to C}{A \to C}\ (r\mathrm{HS}) \qquad \frac{A \to (B \to C)}{B \to (A \to C)}\ (rC).$$

The former is derivable from rB and modus ponens, and the latter from (C) and modus ponens.

Applying rC to the Identity instance $(A \to B) \to (A \to B)$, we obtain

$$\vdash A \to ((A \to B) \to B). \tag{1}$$

From (1) we obtain

$$\vdash (((A \to B) \to B) \to C) \to (A \to C), \tag{2}$$
$$\vdash ((A \to A) \to B) \to B, \tag{3}$$
$$\vdash \neg((A \to B) \to B) \to \neg A. \tag{4}$$

Here (2) follows by applying rB' to (1), (3)follows by replacing A with $A \to A$ in (1) and applying modus ponens with the instance $A \to A$ of (Id), and (4) follows by rCont.

Replacing A by $\neg A$ in (4) gives $\neg((\neg A \to B) \to B) \to \neg\neg A$. Together with the instance $\neg\neg A \to A$ of (DNe), an application of rHS yields

$$\vdash \neg((\neg A \to B) \to B) \to A. \tag{5}$$

Applying rB' to the two Boethius axioms gives

$$\vdash (\neg(A \to \neg B) \to C) \to ((A \to B) \to C), \tag{6}$$
$$\vdash (\neg(A \to B) \to C) \to ((A \to \neg B) \to C). \tag{7}$$

From (4)–(7), we can derive the following:

$$\vdash ((A \to \neg B) \to B) \to \neg A, \tag{8}$$
$$\vdash ((A \to B) \to \neg B) \to \neg A, \tag{9}$$
$$\vdash ((\neg A \to B) \to \neg B) \to A. \tag{10}$$

To obtain (8), replace A by $A \to \neg B$ and C by $\neg A$ in (6). This gives

$$\big(\neg((A \to \neg B) \to \neg B) \to \neg A\big) \to \big(((A \to \neg B) \to B) \to \neg A\big).$$

The antecedent is the instance of (4) obtained by replacing B with $\neg B$, so modus ponens yields (8). To obtain (9), replace A by $A \to B$ and C by $\neg A$ in (7). This gives

$$\big(\neg((A \to B) \to B) \to \neg A\big) \to \big(((A \to B) \to \neg B) \to \neg A\big).$$

Its antecedent is exactly (4), so modus ponens yields (9). Finally, to obtain (10), replace A by $\neg A \to B$ and C by A in (7). This gives

$$\big(\neg((\neg A \to B) \to B) \to A\big) \to \big(((\neg A \to B) \to \neg B) \to A\big).$$

Its antecedent is exactly (5), so modus ponens yields (10).

Composing the antecedent of each of (8)–(10) with an instance of (B') and then applying rB twice gives

$$\vdash ((A \to \neg B) \to C) \to ((C \to B) \to \neg A), \tag{11}$$
$$\vdash ((A \to B) \to C) \to ((C \to \neg B) \to \neg A), \tag{12}$$
$$\vdash ((\neg A \to B) \to C) \to ((C \to \neg B) \to A). \tag{13}$$

For (11), begin with (8):

$$((A \to \neg B) \to B) \to \neg A.$$

Prefixing by $C \to B$ using rB gives

$$\big((C \to B) \to ((A \to \neg B) \to B)\big) \to \big((C \to B) \to \neg A\big).$$

Prefixing once more, now by $(A \to \neg B) \to C$, gives

$$\Big(((A \to \neg B) \to C) \to ((C \to B) \to ((A \to \neg B) \to B))\Big)$$
$$\to \Big(((A \to \neg B) \to C) \to ((C \to B) \to \neg A)\Big).$$

The antecedent is the following instance of (B'):

$$((A \to \neg B) \to C) \to ((C \to B) \to ((A \to \neg B) \to B)).$$

Modus ponens therefore yields (11). Similarly, for (12), begin with (9):

$$((A \to B) \to \neg B) \to \neg A.$$

Two applications of rB, first prefixing by $C \to \neg B$ and then by $(A \to B) \to C$, give

$$\Big(((A \to B) \to C) \to ((C \to \neg B) \to ((A \to B) \to \neg B))\Big)$$
$$\to \Big(((A \to B) \to C) \to ((C \to \neg B) \to \neg A)\Big).$$

The antecedent is the following instance of (B'):

$$((A \to B) \to C) \to ((C \to \neg B) \to ((A \to B) \to \neg B)).$$

Modus ponens therefore yields (12). Finally, for (13), begin with (10):

$$((\neg A \to B) \to \neg B) \to A.$$

Two applications of rB, first prefixing by $C \to \neg B$ and then by $(\neg A \to B) \to C$, give

$$\Big(((\neg A \to B) \to C) \to ((C \to \neg B) \to ((\neg A \to B) \to \neg B))\Big)$$
$$\to \Big(((\neg A \to B) \to C) \to ((C \to \neg B) \to A)\Big).$$

The antecedent is the following instance of (B'):

$$((\neg A \to B) \to C) \to ((C \to \neg B) \to ((\neg A \to B) \to \neg B)).$$

Modus ponens therefore yields (13).

From (11), (12), and (2), we can derive:

$$\vdash A \to ((\neg B \to B) \to \neg A), \tag{14}$$
$$\vdash A \to ((B \to \neg B) \to \neg A). \tag{15}$$

For (14), applying the substitution $C \mapsto \neg B$ in (11) gives

$$((A \to \neg B) \to \neg B) \to ((\neg B \to B) \to \neg A).$$

Now replace B by $\neg B$ and C by $(\neg B \to B) \to \neg A$ in (2). The resulting instance is

$$\Big(((A \to \neg B) \to \neg B) \to ((\neg B \to B) \to \neg A)\Big)$$
$$\to \Big(A \to ((\neg B \to B) \to \neg A)\Big).$$

Its antecedent is the displayed instance of (11), so modus ponens yields (14). Similarly, for (15), applying the substitution $C \mapsto B$ in (12) gives

$$((A \to B) \to B) \to ((B \to \neg B) \to \neg A).$$

Replace C in (2) by $(B \to \neg B) \to \neg A$. This gives

$$\Big(((A \to B) \to B) \to ((B \to \neg B) \to \neg A)\Big)$$
$$\to \Big(A \to ((B \to \neg B) \to \neg A)\Big).$$

Its antecedent is the displayed instance of (12), so modus ponens yields (15).

Exchange applied to (14) gives

$$\vdash (\neg B \to B) \to (A \to \neg A). \tag{16}$$

Replacing A by p and B by q in (16) gives

$$\vdash (\neg q \to q) \to (p \to \neg p).$$

Since $\mathrm{Var}(\neg q \to q) = \{q\}$ and $\mathrm{Var}(p \to \neg p) = \{p\}$, this theorem violates variable sharing. $\square$

Now, Tedder observes that his finite matrices validate $(A \to A) \to (B \to B)$, but leaves open whether this formula is derivable. We show that it is.

Theorem 2. *If p and q are distinct propositional variables, then*

$$\vdash_{\mathbf{RWB}^{-}_{\to,\neg}} (p \to p) \to (q \to q).$$

Proof. From (15) and (3), we derive

$$\vdash (A \to \neg A) \to \neg(B \to B). \tag{17}$$

The relevant instance of (15) is obtained by the following substitution:

$$A \mapsto B \to B, \qquad B \mapsto A$$

which gives us

$$(B \to B) \to \big((A \to \neg A) \to \neg(B \to B)\big).$$

The relevant instance of (3) is obtained by the following substitution:

$$A \mapsto B, \qquad B \mapsto (A \to \neg A) \to \neg(B \to B).$$

which gives us

$$\Big((B \to B) \to \big((A \to \neg A) \to \neg(B \to B)\big)\Big) \\ \to \Big((A \to \neg A) \to \neg(B \to B)\Big).$$

Modus ponens therefore yields (17) (and another failure of variable sharing).

Finally, from (16) and (13), we get:

$$\vdash ((A \to \neg A) \to \neg B) \to B. \tag{18}$$

The relevant instance of (13) is obtained by the following substitution:

$$A \mapsto B, \qquad B \mapsto B, \qquad C \mapsto A \to \neg A.$$

This gives

$$\Big((\neg B \to B) \to (A \to \neg A)\Big) \to \Big(((A \to \neg A) \to \neg B) \to B\Big).$$

Its antecedent is exactly (16). Modus ponens therefore yields (18).

We now construct a bridge between arbitrary identity formulas. Taking the instance

$$((C \to C) \to B) \to B$$

of (3), applying contraposition, and then prefixing by A gives

$$\vdash (A \to \neg B) \to \big(A \to \neg((C \to C) \to B)\big). \tag{19}$$

From (19) and (1), we get:

$$\vdash (A \to \neg B) \to \Big(((A \to \neg((C \to C) \to B)) \to D) \to D\Big). \tag{20}$$

The relevant instance of (1) is obtained by the following substitution:

$$A \mapsto A \to \neg((C \to C) \to B), \qquad B \mapsto D$$

This gives

$$(A \to \neg((C \to C) \to B)) \to \Big(((A \to \neg((C \to C) \to B)) \to D) \to D\Big)$$

The consequent of (19) is the antecedent of this instance of (1). An application of rHS therefore yields (20).

From (17) and (20), we obtain

$$\vdash \Big(((A \to \neg A) \to \neg((C \to C) \to (B \to B))) \to D\Big) \to D. \tag{21}$$

The relevant instance of (20) is obtained by the substitution:

$$A \mapsto A \to \neg A, \qquad B \mapsto B \to B$$

The resulting instance is

$$\begin{aligned}&((A \to \neg A) \to \neg(B \to B))\\ &\quad \to \Big(((A \to \neg A) \to \neg((C \to C) \to (B \to B))) \to D\Big) \to D.\end{aligned}$$

The antecedent is exactly (17). Modus ponens therefore yields (21).

Now, let

$$T = (p \to p) \to (q \to q).$$

Putting $C = p$, $B = q$, and $D = T$ in (21) gives

$$\vdash (((A \to \neg A) \to \neg T) \to T) \to T. \tag{22}$$

But the instance of (18) obtained by replacing B with T is

$$\vdash ((A \to \neg A) \to \neg T) \to T. \tag{23}$$

Modus ponens on (22) and (23) yields $\vdash T$. That is,

$$\vdash (p \to p) \to (q \to q).$$

□

References

[1] Andrew Tedder. Negated implications in connexive relevant logics. *Australasian Journal of Logic*, 22(1), Article 2, 2025.