There is a Logical Negation: “Yes,” “No,” Both, and Neither

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Abstract

Jc Beall agues that if FDE is logic proper, then there is no logical negation. This claim is largely based on the fact that, in standard proof systems for FDE, there are no stand-alone negation rules that suffice to capture the behavior of negation. In this paper, I show that by adopting a *bilateral* proof system for FDE, one can maintain that there is a logical negation, it is the very same logical negation that belongs to classical logic, and its basic function is to flip-flop between assertion and denial. I conclude by developing the conception of assertion and denial on which this proposal is based, responding to a number of objections.

Key Words: Negation; Bilateralism; Non-Classical Logic; Proof-Theoretic Semantics; FDE

0 Introduction

In his paper, “There is No Logical Negation: True, False, Both, Neither,” Jc Beall argues, first, that there are reasons to adopt the weak subclassical logic FDE as “logic proper,” and, second, that if we do, there is no logical negation. I focus here on the second, conditional claim, granting, at least for the sake of argument, the first claim (though I am indeed sympathetic to it). Though not explicitly articulated in this way, Beall’s conclusion about the lack of logical negation can be understood as motivated in broadly proof-theoretic terms. In particular, in standard proof systems for FDE, there are no separable negation rules that characterize the inferential behavior of negation, nor (as in the case of LP or K3) are there axiom schemas involving negation as the sole logical operator. In this paper, I show that, by adopting a bilateral approach to FDE, adopting a proof system in which formulas are positively and negatively signed to express assertion and denial, Beall’s reasons for thinking that there is no logical negation vanish. There is a logical negation, it’s the very same logical negation that belongs to classical logic, and its basic function is to flip-flop between assertion and denial. Though this bilateral perspective on FDE is extremely natural, it has not been hitherto adopted. The reason for this, I believe, is that the conception of assertion and denial on which it is based, according to which these two notions are not necessarily incompatible, has seemed to many to be a theoretical non-starter. I thus conclude by developing this conception in detail, the core idea being to distinguish between *two orthogonal dimensions* of bilateralism: *reasons for* and *reasons against*, on the one hand, and *assertion* and *denial*, on the other. Making this conception explicit enables me to respond to a number of objections.

1 FDE as "Logic Proper"

Let me start by briefly laying out the motivation for taking FDE to be "logic proper." In a first logic course, one learns the truth and falsity conditions for logical connectives of negation, conjunction, and disjunction.[1] A negation is true just in case the negatum is false, and a negation is false just in case the negatum is true. Likewise, a conjunction is true just in case both conjuncts are true, and a conjunction is false just in case at least one of the conjuncts is false. Dually for disjunction. Officially, where 1 is truth and 0 is falsity, the truth and falsity conditions for the standard logical connectives are given as follows:

$$v(\neg A) = \begin{cases} 1, & \text{if } v(A) = 0 \\ 0, & \text{if } v(A) = 1 \end{cases}$$

$$v(A \wedge B) = \begin{cases} 1, & \text{if } v(A) = 1 \text{ and } v(B) = 1 \\ 0, & \text{if } v(A) = 0 \text{ or } v(B) = 0 \end{cases}$$

$$v(A \vee B) = \begin{cases} 1, & \text{if } v(A) = 1 \text{ or } v(B) = 1 \\ 0, & \text{if } v(A) = 0 \text{ and } v(B) = 0 \end{cases}$$

In the context of classical logic, we assume that truth and falsity are exclusive and exhaustive, such that no sentence can be both true and false and no sentence can be neither true nor false. These assumptions, however, seem to be *substantive* ones, and both of them have been called into question in certain contexts, the most famous of which are those pertaining to paradoxes such as the liar. Regardless of what one ultimately wants to say about paradoxes such as the liar, it seems clear that logic enables us to investigate the consequences of all of the things that one *could* say, where, among logically possible options, are ones that reject exclusivity or exhaustivity. Though there may be compelling reasons to accept exclusivity and/or exhaustivity when dealing with paradoxes such as the liar, these aren't strictly speaking *logical* reasons; *logic itself* doesn't force us into such an acceptance.

If one is moved by considerations of the above sort, then one will think that, as far as logic itself is concerned, we can allow that sentences may have one of four possible valuations: just true (or $\{1\}$), just false (or $\{0\}$), both true and false (or $\{1, 0\}$), or neither true nor false (or $\varnothing$). Adopting this more permissive conception of what is

[1] I ignore the material conditional here, treating it as defined in terms of these connectives.

logically possible, we can maintain that the semantic clauses for logical connectives are just those stated above; we simply swap the "=" sign (the use of which involves the assumption of classicality) with the "∋" sign (the use of which does not involve this assumption):

$$v(\neg A) \ni \begin{cases} 1, & \text{if } v(A) \ni 0 \\ 0, & \text{if } v(A) \ni 1 \end{cases}$$

$$v(A \wedge B) \ni \begin{cases} 1, & \text{if } v(A) \ni 1 \text{ and } v(B) \ni 1 \\ 0, & \text{if } v(A) \ni 0 \text{ or } v(B) \ni 0 \end{cases}$$

$$v(A \vee B) \ni \begin{cases} 1, & \text{if } v(A) \ni 1 \text{ or } v(B) \ni 1 \\ 0, & \text{if } v(A) \ni 0 \text{ and } v(A) \ni B \end{cases}$$

Relaxing things in this way, and defining validity as preservation of truth in all valuations, we obtain the logic known as FDE (first-degree entailment). Excluding $\varnothing$ from the set of admissible valuations, we get LP. Excluding $\{1,0\}$, we get K3.[2] Excluding both $\varnothing$ and $\{1,0\}$, we get Classical Logic. Insofar as logical possibility is maximally broad, it is natural to conceive of FDE as telling us what is logically possible, and stronger logics such as LP, K3, and CL as resulting from excluding certain logical possibilities from consideration. While such an exclusion of possibilities is of course justified in many contexts, this justification is not *logical* justification

2 Unilateral Proof Systems for FDE (and LP, K3, and CL)

Having briefly stated the motivating idea for taking FDE to be "logic proper," let me turn to Beall's claim that, on this conception of logic, "there is no logical negation." Before providing the positive argument that there is logical negation, even on FDE, let me first just say why this is a *prima facie* puzzling thing for Beall to say, if we just look at the semantics stated above. A core idea of retaining the semantic clauses stated above seems to be that the logical connectives we have in FDE are just those that we have in CL. After all, they have the very same semantic clauses. It's just that these semantic clauses operate on a broader space of possibilities for the truth-values of sentences. So, it seems natural to say that, just as logical conjunction is just what it is on the classical conception, so too logical negation is just what it is on the classical conception. Why, then, does Beall conclude that there is no logical negation? The answer, I think, has to do not with the *semantics* of FDE stated above, but with the standard *proof systems* for FDE.

[2]FDE is Anderson and Belnap's [1] logic of "First Degree Entailment." LP is Preist's [26] "Logic of Paradox," first proposed by Asenjo [2]. K3 is most famously deployed by Kripke [21] (see also Kremer [19]). For an introductory overview of these logics, see Beall, Glanzberg and Ripley [6], Chapter 5.

In the context of this paper, I will follow Beall [4] [5] and focus my discussion on the *sequent calculus* presentation of FDE and related logics. All of the points I will be making in what follows can be made in exactly the same way in the context of *natural deduction* systems for these logics, but I will not pursue these points in those terms here.[3] Let us start with the classical sequent calculus[4]

$$\frac{}{X, A \vdash A, Y}\text{ Reflex.}$$

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\neg_{\text{L}} \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\neg_{\text{R}}$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\wedge_{\text{L}} \qquad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\wedge_{\text{R}}$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y}\vee_{\text{L}} \qquad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\vee_{\text{R}}$$

The negation rules, in particular, are part of what give the classical sequent calculus a formal elegance that is not possessed by its natural deduction counterpart. Whereas standard natural deduction systems for classical logic famously lack harmonious negation rules, in the sequent calculus, classical negation is codified by above rules enabling one to "flip-flop" a sentence across the turnstile. It is easy to see how having these rules amounts to imposing Explosion and Excluded Middle. From the axiom of Reflexivity, we have $A \vdash A, B$, and so the left rule enables us to derive $A, \neg A \vdash B$ for any sentence B. Likewise, from Reflexivity, the right rule enables one to derive $\vdash A, \neg A$. Thus, if one is putting forward a sequent calculus for FDE, which enables one to derive neither explosion nor Excluded Middle, one must reject both such negation rules.

Beall [3] puts forward the following sequent calculus for FDE, based on Priest's [29] tableau system:

$$\frac{}{X, A \vdash A, Y}\text{ Reflex.}$$

$$\frac{X \vdash A, Y}{X \vdash \neg\neg A, Y}\neg\neg_{R} \qquad \frac{X, A \vdash Y}{X, \neg\neg A \vdash Y}\neg\neg_{L}$$

[3]See [omitted] for bilateral natural deduction systems for the FDE family. The operational rules of these systems are precisely those of (implication-free fragment of) Rumfitt's [33] classical system that Kürbis [22] calls "$\mathfrak{B}$." As with the sequent calculi put forward in what follows here, all that changes between the systems are the coordination principles (in particular, the versions of Bilateral Explosion and Excluded Middle considered by del Valle-Inclan [9]).

[4]This, in particular, is the version of the classical sequent calculus proposed by [18], which has many nice proof-theoretic properties. See [24] for an overview.

$$\frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y} \wedge_R \qquad \frac{X \vdash \neg A, \neg B, Y}{X \vdash \neg(A \wedge B), Y} \neg\wedge_R$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y} \wedge_L \qquad \frac{X, \neg A \vdash Y \quad X, \neg B \vdash Y}{X, \neg(A \wedge B) \vdash Y} \neg\wedge_L$$

$$\frac{X \vdash A, B}{X \vdash A \vee B} \vee_R \qquad \frac{X \vdash \neg A, Y \quad X \vdash \neg B, Y}{X \vdash \neg(A \vee B), Y} \neg\vee_R$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y} \vee_L \qquad \frac{X, \neg A, \neg B \vdash Y}{X, \neg(A \vee B) \vdash Y} \neg\vee_L$$

Notably, this sequent calculus features not only the standard conjunction and disjunction rules, familiar from the classical sequent calculus, but also rules for *negated* conjunctions and disjunctions. For LP, one adds (the multiple conclusion generalization of) Excluded Middle, $X \vdash A, \neg A, Y$ and, for K3, one adds (the multiple conclusion generalization) Explosion, $X, A, \neg A \vdash Y$. Adding both, we get classical logic. Equivalently, for LP one can add classical logic's right negation rule, and, for K3, one can add classical logic's left negation rule.[5] Adding both, of course, gives us classical logic, and, if we do have both then we can get rid of the negated conjunction and disjunction rules, giving us the familiar classical sequent calculus.

If we look at this proof system, it seems that there is a fundamental difference in the treatment of conjunction and disjunction, on the one hand, and the treatment of negation, on the other. On the one hand, it contains the classical rules for conjunction and disjunction shown above. On the other hand, it *can't* contain the classical rules for negation without collapsing into the classical sequent calculus. Accordingly the negation rules must be given in a different way, and, crucially, unlike the classical sequent calculus, there are no separable rules for negation that suffice to characterize the distinctive behavior of negation. There are, of course, *double* negation rules, but these rules only suffice to tell us that negation is an involution, but that, of course, is true as well of the operator Beall calls "logical nullation," expressed in English by "It's true that." Thus, in order to classify the inferential behavior of negation, this sequent calculus relies on rules that characterize negation's interaction with the other logical connectives. Now, in LP and K3 there are at least *some* rules that characterize the stand-alone behavior of negation: the axioms of Excluded Middle in LP and Explosion in K3, or, equivalently, $\neg_R$ in LP and $\neg_L$ in K3. In FDE, however, there are *no* such rules. Thus Beall [5], taking FDE to be logic proper, concludes that "there is no logical negation," (15).

[5] Proof is straightforward. Consider just the case of LP. We know adding Excluded Middle yields LP, so for completeness, just note that Excluded Middle is immediately derivable from Reflexivity and $\neg R$. For soundness, it is straightforward to show the admissibility of $\neg R$ in the system for LP with Excluded Middle by induction on proof height.

3 Bilateralist Negation

What is the negation operator that, supposedly, the classical logician has but the proponent of FDE does not? Whatever it is, it must be characterized by the standard sequent rules of the classical sequent calculus. Once again, they are the following:

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg\mathrm{L} \qquad\qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \neg\mathrm{R}$$

But what do these rules actually *say*? In the context of the *logical inferentialist* semantic program [13] [8] [39], which takes seriously the idea that the meaning of a logical connective is given by the inferential rules governing its use as codified by a formal proof system, one cannot simply appeal to the validity of these rules relative to classical semantics to justify them. Rather, they must be straightforwardly intelligible as formally codifying inferential norms. In this context, it has been argued that there is a fundamental issue with appealing to multiple conclusion sequent systems: multiple conclusion "arguments," where the premises are collected conjunctively and the conclusions are collected disjunctively, don't seem to correspond to anything in our ordinary inferential practices.[6] In response to this sort of concern, Restall [30] proposes a reading of multiple conclusion sequents, according to which $X \vdash Y$ is understood as saying that *asserting* everything in X along with *denying* everything in Y is incoherent or "out of bounds." Thus, the turnstile is not, in the first instance, playing the role of separating *premises* from *conclusions*, but, rather, of separating *assertions* from *denials*. Reading sequents in this fashion, the left rule says that if, relative to any position consisting in asserting everything in X and denying everything in Y, denying A is out of bounds, then, relative to that position, asserting $\neg A$ is out of bounds. Likewise, the right rule says that if, relative to any position, asserting A is out of bounds, then, relative to that position, denying $\neg A$ is out of bounds. Thus, on this conception, negation *is* a flip-flop operator, but what it's really flipping and flopping between is assertion and denial.

This basic "bilateralist" conception of the function of classical negation as flipping between assertion and denial has been defended in a different formal context by Rumfitt [33], drawing on prior work from Smiley [36]. Rather than using bilateralism to *interpret* Gentzen's multiple conclusion sequent calculus, Rumfitt introduces signs "+" and "−" for assertion and denial to *bilateralize* Gentzen's natural deduction system for classical logic so as to be able to provide harmonious rules for negation. The rules of Rumfitt's system (formulated in "logistic" notation) are the following:

$$\frac{\Gamma \vdash -\langle A\rangle}{\Gamma \vdash +\langle \neg A\rangle}\ +_{\neg} \qquad\qquad \frac{\Gamma \vdash +\langle A\rangle}{\Gamma \vdash -\langle \neg A\rangle}\ -_{\neg}$$

Reading the turnstile as expressing a relation of committive consequence, these rules can be understood as saying that one is committed to asserting $\neg A$ just in case one

[6] See [38] for a sustained statement of this problem.

is committed to denying A and one is committed to denying $\neg A$ just in case one is committed to asserting A.

These two bilateral conceptions of classical negation are obviously quite close, and it is natural to wonder about the relation between them. In fact, the two conceptions can be formally brought together by transposing the multiple conclusion sequent calculus, as interpreted by Restall, into the sort of signed notation proposed by Rumfitt. The basic idea is this: in a unilateral sequent calculus, a formula of the form $X \vdash$ can be understood as expressing that all of the sentences in X are jointly inconsistent. In a *bilateral* sequent calculus, then, we might take a formula of the form $\Gamma \vdash$, where Γ is a set of signed formulas, to express the same thing: that the set of moves in Γ, be they assertions or denials, are incoherent. This suggests the following translation of multiple conclusion unilateral sequents, on Restall's interpretation, into solely left-sided bilateral sequents, and vice versa:

> **Translation Schema:** To translate an unsigned multiple conclusion sequent of the form $X \vdash Y$ to a signed sequent of the form $\Gamma \vdash$, let $\Gamma = \{+\langle A\rangle \mid A \in X\} \cup \{-\langle B\rangle \mid B \in Y\}$. Conversely, to translate a signed sequent of the form $\Gamma \vdash$ to an unsigned multiple conclusion sequent of the form $X \vdash Y$, let $X = \{A \mid +\langle A\rangle \in \Gamma\}$ and $Y = \{B \mid -\langle B\rangle \in \Gamma\}$.

Translating multiple conclusion sequents in this way, the classical negation rules come out as follows:

$$\frac{\Gamma, -\langle A\rangle \vdash}{\Gamma, +\langle \neg A\rangle \vdash}\ +_{\neg} \qquad\qquad \frac{\Gamma, +\langle A\rangle \vdash}{\Gamma, -\langle \neg A\rangle \vdash}\ -_{\neg}$$

Translated in this way, these two bilateralist conceptions of negation (Rumfitt's, understood in terms of committive consequence, and Restall's, understood in terms of normative incoherence) collapse into one just in case we impose certain *coordination principles*, bilateral structural rules which "coordinate" the opposite speech acts of assertion and denial. In particular, where φ is a signed formula (expressing the assertion or denial of some sentence) and φ^* is the oppositely signed formula (expressing the denial or assertion of that sentence), the coordination principles that collapse the two negation rules into one might be most perspicuously stated as follows:

$$\frac{\Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash}\ \text{In} \qquad\qquad \frac{\Gamma, \varphi \vdash}{\Gamma \vdash \varphi^*}\ \text{Out}$$

In says that if Γ *commits* one to φ, then Γ along with φ^* is *incoherent*, whereas Out says that if Γ along with φ is *incoherent*, then Γ *commits* one to φ^*. If we *don't* impose such coordination principles, then we need both pairs of negation rules.

Generalizing, we might think of the multiple conclusion negation rules, understood in Restall-style bilateralist fashion, as specifying a particular case of the *premisory* role of asserting or denying a negation (the case in which there is a null set of conclusions), whereas Rumfitt's bilateral rule specify the *conclusory* role of asserting or denying a negation. Putting these two sets of rules together, then, we have the following set of rules:

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta}{\Gamma, +\langle \neg A\rangle \vdash \Delta}\,+_{\neg L} \qquad \frac{\Gamma, +\langle A\rangle \vdash \Delta}{\Gamma, -\langle \neg A\rangle \vdash \Delta}\,-_{\neg L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta}{\Gamma \vdash +\langle \neg A\rangle, \Delta}\,+_{\neg R} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta}{\Gamma \vdash -\langle \neg A\rangle, \Delta}\,-_{\neg R}$$

These rules tell us that asserting a negation has the same role, as either a premise or conclusion, as denying the negatum, and denying a negation has the same role as asserting the negatum. If the bilateralist story about negation is right, then these rules inferentially specify the meaning of negation.[7] Now, if one has (the multiple conclusion generalizations of) In and Out, then, in the context of a multiple conclusion bilateral system, half of these rules are redundant; one can take just the left rules or just the right rules (or, indeed, one can just take the familiar classical negation rules as positive rules). However, if we are inclined to treat FDE as logic proper for the reasons articulated above, we cannot accept In and Out. Consider, for instance, that, given Reflexivity, we have $+\langle p\rangle \vdash +\langle p\rangle, +\langle q\rangle$. Given In, we can conclude $+\langle p\rangle, -\langle p\rangle \vdash +\langle q\rangle$, an explosion principle which says that asserting and denying some sentence p commits one to asserting an arbitrary sentence q. This should not be accepted by the bilateralist proponent of FDE who accepts gluts, thinking that some sentences are both true and false. Insofar as assertion just is a speech act in which one commits oneself to the truth of a sentence and denial is a speech act in which one commits oneself to the falsity of a sentence, accepting gluts amounts to accepting that some sentence are such that they are both to-be-asserted and to-be-denied. However, asserting and denying some sentence should not commit one to asserting everything. Thus, we must reject In. Analogous gappy reasoning applies to the rejection of Out. Thus, for a proponent of FDE, all four negation rules, codifying negation's role as flip-flopping between assertions and denials as both premises and conclusions, are necessary.

This move to a bilateral setting should come as extremely natural to the sub-classical logician, and especially the maximally sub-classical proponent of FDE. The crucial thought of FDE in the context of the *semantics* is that we must treat truth and falsity conditions as on a par. We cannot reduce falsity conditions to truth conditions, as one can in the context of classical logic where falsity simply aligns with untruth. The corresponding thought in the context of the bilateral *proof-theory* is that we must treat assertion conditions and denial conditions as on a par. We cannot reduce denial

[7] Specifying the role of a sentence containing a logical operator as both a premise and a conclusion with a sequent calculus containing left and right rules is one way of providing a "two-aspect model of meaning" of the sort associated with Dummett [10], with these rules respectively codifying the inferential *consequences* and *conditions* of the use of that sentence. See [20] for an account of how the sequent calculus can be understood as providing an inferentialist theory of logical content in this way. I will return to consider questions concerning the issue of providing an inferentialist account of negation at the end of this paper.

conditions to assertion conditions, as one can in the context of classical logic, where the correctness of denial simply aligns with the incorrectness of assertion. Let me now formulate bilateral proof systems for the FDE family containing these bilateral negation rules.

4 Bilateral Proof Systems for FDE (and LP, K3, and CL)

We start by extending the familiar *unilateral* notion of validity, understood as preservation of *truth*, to a *bilateral* notion of validity, understood as preservation of *correctness*. Officially, the correctness of an assertion or denial is defined as follows:

> **Correctness:** Asserting A is *correct*, relative to some valuation v, just in case $1 \in v(A)$. Denying A is *correct*, relative to v, just in case $0 \in v(A)$.

We now define bilateral validity as follows:

> **Bilateral Validity:** An argument of the form $\Gamma \vdash \Delta$ is *bilaterally valid*, relative to a set of admissible valuations V, $\Gamma \vDash_{B_V} \Delta$, just in case there is no $v \in V$ such that all of the stances in Γ are correct and all of the stances in Δ are incorrect.

In this way, we extend the familiar *unilateral* consequence relations of FDE, LP, K3 to *bilateral* consequence relations. The following sequent calculus is sound and complete relative to the bilateral consequence relation of FDE:

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta}\ \text{Reflex.}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta}{\Gamma, +\langle \neg A\rangle \vdash \Delta}\ +_{\neg_L} \qquad \frac{\Gamma, +\langle A\rangle \vdash \Delta}{\Gamma, -\langle \neg A\rangle \vdash \Delta}\ -_{\neg_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta}{\Gamma \vdash +\langle \neg A\rangle, \Delta}\ +_{\neg_R} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta}{\Gamma \vdash -\langle \neg A\rangle, \Delta}\ -_{\neg_R}$$

$$\frac{\Gamma, +\langle A\rangle, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \wedge B\rangle \vdash \Delta}\ +_{\wedge_L} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta \quad \Gamma \vdash +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \wedge B\rangle, \Delta}\ +_{\wedge_R}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta \quad \Gamma, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \wedge B\rangle \vdash \Delta}\ -_{\wedge_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \wedge B\rangle, \Delta}\ -_{\wedge_R}$$

$$\frac{\Gamma, +\langle A\rangle \vdash \Delta \quad \Gamma, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \vee B\rangle \vdash \Delta}\ +_{\vee_L} \qquad \frac{\Gamma \vdash +\langle A\rangle, +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \vee B\rangle, \Delta}\ +_{\vee_R}$$

$$\frac{\Gamma, -\langle A\rangle, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \vee B\rangle \vdash \Delta}\ -_{\vee_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta \quad \Gamma \vdash -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \vee B\rangle, \Delta}\ -_{\vee_R}$$

For Bilateral LP, we add the principle of *Bilateral Excluded Middle*, $\Gamma \vdash \varphi, \varphi^*, \Delta$, and, for Bilateral K3, we add *Bilateral Explosion*, $\Gamma, \varphi, \varphi^* \vdash \Delta$. Adding both, we get bilateral classical logic.[8] Equivalently, for LP, we can add (the multiple conclusion generalization of) Out, and, for K3, we can add (the multiple conclusion generalization of) In, and, once again, adding both, we get classical logic.[9] This bilateral proof system is obviously quite close to the unilateral one shown above. However, there are three crucial points about this system that deserve emphasis.

The first crucial point is that all of the rules in this system are *separable*. The inferential behavior of negation is given only by the negation rules, not by rules codifying its interaction with other connectives. Moreover, adding any set of rules to the fragment of the sequent calculus not containing those rules constitutes a conservative extension.[10] Separability is widely taken to be key formal constraint, on a par with harmony, in the context of logical inferentialism.[11] If rules are *not* separable—if, for instance, the rules for conjunction are not separable from the rules for negation—then, if we take seriously the that idea knowing the meaning of a connective is mastering the rules governing its use, it would seem that one could not know the meaning of negation without knowing the meaning of conjunction, nor could one know the meaning of conjunction without knowing the meaning of negation. As I explained above, the lack of separable rules for the negation

[8]One might wonder how these facts relate to Restall's style bilateralism discussed above, where, dropping the exclusivity or exhaustivity of assertion and denial results in substructural logics rather than subclassical logics. In particular, in Restall's bilateralist setting, dropping the exclusivity of assertion and denial amounts to rejecting Reflexivity, yielding the substructural logic TS, whereas dropping the exhaustivity of assertion and denial amounts to rejecting Cut, yielding the substructural logic ST. Note, however, that these results in Restall's setting concern *unilateral consequence interpreted bilaterally*, whereas here we are working *bilateral consequence*. The logics discussed here—Bilateral LP and Bilateral K3—are so-called because their *positively signed fragments* aligns with the familiar consequence relations of LP and K3. That is, considering only sequents with only positively signed formulas, $+\langle X\rangle \vdash_{\mathrm{BLP}} +\langle Y\rangle$ just in case $X \vdash_{\mathrm{LP}} Y$, and likewise for K3. Note, however, we can also consider the relation between the *solely left-sided fragment* of a bilateral consequence relation and a unilateral consequence relation, mapping $+\langle X\rangle, -\langle Y\rangle \vdash$ onto $X \vdash Y$, with the thought that $X \vdash Y$ can be understood as saying that asserting everything in X and denying everything in Y is incoherent. Given this translation schema and definition of bilateral validity provided, it is easy to see that the solely left-sided fragment of Bilateral K3 *just is* ST, with its bilateral interpretation made explicit, whereas the solely left-sided fragment of Bilateral LP *just is* TS. Moveover, if one just looks at the solely left-sided fragment of this sequent calculus (just the set of left rules where Δ is null), under this translation schema, this *just is* the classical sequent calculus shown above (the axiom of Reflexivity corresponding to (solely left-sided) Bilateral Explosion, which we have in BK3 but not BLP). Thus, these bilateral subclassical logics actually include (the translations of) familiar substructural logics. For these results and a discussion of their philosophical consequences, particularly concerning the "substructural" approach to paradox developed by Ripley [31], see [reference omitted].

[9]It should be easy to see that these claims are true. However, the full proofs are provided in [reference omitted]. The admissibility of Cut, Weakening, and the eliminability of non-atomic instances of Reflexivity are also proven for these systems there.

[10]This is notably not the case for the negation rules of standard unilateral natural deduction systems for classical logic, as evidenced by tautologies such as Peirce's law, which contains only the conditional yet is not provable in negation-free implicative fragment, given the usual conditional rules. In the sequent calculus setting, conservativity is proven by the proof of the admissibility of Cut.

[11]For a recent discussion of separability in the context of logical inferentialism, see [23].

in standard proof systems for FDE is, I think, the main reason that leads Beall to his conclusion that there is no logical negation. This system, in which there are separable rules that precisely characterize the inferential behavior of negation, completely undercuts that reason.

The second closely related crucial point is that the bilateral principles that, added to the sequent calculus for FDE, yield LP or K3 are *not* negation rules. They are, once again, *coordination principles*, bilateral structural rules that "coordinate" the relation between the speech acts of assertion and denial. Beall's thought, transferred into this setting, is that, insofar as logic is maximally topic-neutral, and, in paradoxical contexts these coordination principles can be called into question, *logic itself* does not impose such coordination. At least for the sake of the present paper, I will grant this thought. But to say this is to say *nothing* about negation, since the crucial bilateralist thought is that coordination principles are *not* negation rules; they are distinctively bilateral *structural* rules. Explosion and Excluded Middle can, of course, be expressed with negation. For instance, we can express Explosion using negation as $+\langle A\rangle, +\langle\neg A\rangle \vdash +\langle B\rangle$. However, to think that, because of this it should be understood, fundamentally, as a principle about *negation* is a mistake. For instance, analogously, just because we can express Explosion as $+\langle A \wedge \neg A\rangle \vdash +\langle B\rangle$ does not mean that it's a principle about *conjunction*. Just as it's a mistake to talk about the distinction between "LP conjunction" and "Classical conjunction" on the basis that LP rejects this principle and Classical Logic accepts it, so too is it a mistake to talk about the distinction between "LP negation" and "Classical negation."

This brings us to the final crucial point, which is that, just as the conjunction rules are the same in each logic, so too, the negation rules are *exactly the same* whether one endorses FDE, LP, K3, or CL. Negation is a logical operator such that $\neg A$ is to be asserted just in case A is to be denied, and $\neg A$ is to be denied just in case A is to be asserted. That is what logical negation does; it toggles between a sentence's *truth*, its being correct to *assert*, and a sentence's *falsity*, its being correct to *deny*. I contend, then, that there is a logical negation, and it just is the logical operator that does just that.

5 Two Dimensional Bilateralism

I have laid out bilateral proof systems for the logics in the FDE family according to which the rules for negation are just those of bilateral proof systems for classical logic: negation is an operator such that, for any sentence A, asserting $\neg A$ is inferentially equivalent to denying A, and denying $\neg A$ is inferentially equivalent to asserting A. Thus, insofar as bilateral systems for classical logic can be understood as providing an account of logical negation in terms of its inferential role, these bilateral systems for the logics in the FDE can be understood as showing that the very same negation operator is operative in the context of these subclassical systems. All that differs between these systems is the principles "coordinating" the opposite acts of assertion and denial, establishing them as exhaustive, exclusive, or both. In the remainder of this paper, I want to spell out this conception of bilateralism according to which it makes sense to consider bilateral systems in which coordination principles are relaxed or, indeed, absent entirely. This will enable me to respond to a series of objections for the approach I have laid out.

Let me motivate this discussion by considering what might seem to be the most pressing objection for the bilateral approach that I've laid out, especially from the inferentialist perspective that has motivated previous developments of bilateralism. The objection might run as follows:

> It's crucial to bilateralist accounts of negation that the notion of denial in terms of which negation is understood be intelligible independently of negation. Otherwise, of course, there is no non-circular account of negation. In standard bilateral systems, the coordination principles are what give us a grip on the role of the speech act of denial in terms of its relation to assertion. However, in the base system for FDE, there are no coordination principles. Thus, it's unclear what the notion of denial here is even supposed to be. Minimally, it's hard to see how what you're calling "denial" could really count as *denial* if it's not incompatible with assertion. The incompatibility of assertion and denial is essential to the whole bilateralist idea of tying negation to denial, going back to Price [25]. In your bilateral systems for LP and FDE, however, denying a sentence is not even incompatible with asserting it, as evidenced by the fact that, in these logics, $+A, -A \nvDash$. That is, in this logics, it's not incoherent to assert A and deny A. It's hard to see what the notion of denial you're appealing to here could possibly be if it's not even incompatible with assertion.
>
> This issue is particularly pressing insofar as you take yourself to be providing an *inferentialist* account of negation. You've spoken of assertion and denial as "committing oneself to the truth" and "committing oneself to the falsity" of a sentence. I'll grant the basic bilateralist thought that, contra Frege [12], we can distinguish between the act of committing oneself to the falsity of a sentence and the act of committing oneself to the truth of its negation, even if these acts are inferentially equivalent. Still, these characterizations, which appeal to the alethic notions of truth and falsity, are not immediately available to the inferentialist.[12] An inferentialist, proposing a *use theory of meaning* can only appeal to aspects of the practice in which linguistic expressions are used. For instance, on the sort of inferentialist framework developed by Brandom ([8] [14]), assertion and denial can be understood as opposing "moves" in what Brandom [8] speaks of as "the game of giving and asking for reasons." Assertion is a basic move in the game, and denial is a basic counter-move,

[12]See especially Simonelli [35, 13-16] on this point.

> constituting a *challenge* to an assertion. In response to a challenge, one must *justify* one's assertion by citing *reasons* for it, reasons to which the challenger must respond with reasons of her own. Thus, reasons *for* an assertion of some sentence are essentially reasons *against* a denial that sentence, and vice versa. In this way, assertion and denial seem to be fundamentally normatively incompatible speech acts, each functioning to undercut the entitle to the other. Insofar as that is what assertion and denial are, it seems that it cannot possibly be *correct* to both assert and deny very the same sentence, for, in denying, one would be undercutting the entitlement one has to one's very assertion.

In response to this objection, I'll show how this account of assertion and denial makes perfect sense in the context in the context of the sort of inferentialist framework developed by Brandom.

I should start by acknowledging that, of course, *in normal cases*, denying a sentence is incompatible with asserting it. Still, acknowledging this point does not preclude us from considering odd cases in which (all things considered) reasons *for denying* many not constitute (all things considered) *against asserting*, and vice versa. Indeed, the possibility of such cases is implicit in the version of the inferentialist framework recently put forward by Hlobil and Brandom. In this framework, there are two fundamentally different sorts of reason relations: reasons *for* and reasons *against*. There are also two fundamentally different sorts of speech acts for which there may be reasons for or reasons against: *asserting* and *denying*. In the normal contexts that Hlobil and Brandom consider, reasons *for asserting* completely align with with reasons *against denying*, and reasons *for denying* completely align with reasons *against asserting*. However, Brandom himself at least acknowledges the possibility that, in paradoxical cases, these two dimensions might come apart. He says, for instance, "One might think that it is criterial of paradoxical sentences such as the Liar that subjects end up rationally committed *both* to accepting *and* to rejecting them, or that they are paradigms of sentences rational subjects should endeavor *neither* to accept *nor* reject," (54). So, one might have (all things considered) reasons *for asserting* but also have (all things considered) reasons *for denying*, or one might have (all things considered) reasons *against asserting* but also have (all things considered) reasons *against denying*. This is precisely the pair of possibilities, already implicit in Hlobil and Brandom's distinguishing these two dimensions, that this more flexible bilateral framework enables us to explore.

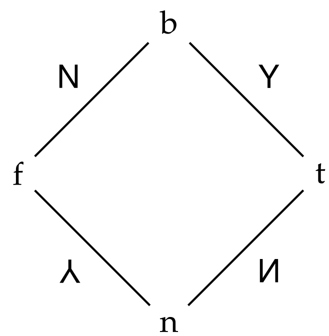
Though Brandom and Hlobil do not pursue this line of thought, the idea of these two dimensions (reasons for/reasons against, on the one hand, and assertion/denial, on the other) coming apart in this way has been explored in a technical semantic context by Blasio, Marcos, and Wansing [7]. I will not go into the technical details of their approach, but, rather, focus on the conceptual underpinnings, which are shared by the proof-theoretic approach developed here.[13] Thinking of these two different

[13] At the most basic level, they consider a semantics for "two-dimensional" consecutions of the form,

dimensions of distinction, we arrive at four different types of reasons relations towards speech acts, which they label in the following way:

	Asserting	**Denying**
Reasons For	Y	N
Reasons Against	⅄	И

The truth-values *true*, *false*, *both*, and *neither*, can be understood as arising at the *intersections* of these four types of reason relations, as depicted in the following diagram:



To explicate this basic conception proof-theoretically, it will be illuminating to make this proof system *doubly bilateral*, such that, in addition to having signs for the acts of *asserting* and *denying*, we also signs for the acts of *making* a discursive move (an act underwritten by *reasons for*) and *challenging* a discursive move (an act underwritten by *reasons against*). Thus, we can distinguish the following four speech acts, each rationally underwritten by the corresponding reason relation stated above:

	the Assertion of A	**the Denial of A**
Making	✓$_+\langle A\rangle$	✓$_-\langle A\rangle$
Challenging	✗$_+\langle A\rangle$	✗$_-\langle A\rangle$

So, ✓$_+\langle A\rangle$ and ✓$_-\langle A\rangle$ express the speech acts of *asserting* A and *denying* A (the acts of *making* these opposite discursive moves), whereas ✗$_+\langle A\rangle$ and ✗$_-\langle A\rangle$ respectively express acts of *challenging* these opposite discursive moves. One can think of the act of challenging a move as an act in which one explicitly registers one's *opposition* to that move.[14] Insofar as both gappy and glutty approaches to paradoxes such as the liar are under consideration, challenging the assertion of some sentence does

$\frac{X_{1,1}}{X_{2,1}} \mid \frac{X_{1,2}}{X_{2,2}}$, which are valid, relative to a set of valuations V, just in case there's no $v \in V$ such that $0 \notin v(A)$ for all $A \in X_{1,1}$, $1 \notin v(B)$ for all $B \in X_{1,2}$, $1 \in v(C)$ for all $C \in X_{2,1}$, and $0 \in v(D)$ for all $D \in X_{2,2}$. It is easy to see that this corresponds to a bilateral sequent of the the form $+\langle X_{2,1}\rangle, -\langle X_{2,2}\rangle \vdash +\langle X_{1,2}\rangle, -\langle X_{1,1}\rangle$.

[14]The acts of challenging an assertion and challenging the denial, expressed here with ✗$_+\langle A\rangle$ and ✗$_-\langle A\rangle$, correspond closely to the acts Incurvati and Schlöder [17] [15] [16] term "weak rejection" and "weak assertion," expressed in their multilateral system with $\ominus\langle A\rangle$ and $\oplus\langle A\rangle$. Inspired by Stalnaker

not logically commit one to denying that sentence nor does denying some sentence logically commit one to challenging the assertion of that sentence. In this doubly bilateral framework, these thoughts can be made formally precise.

Above I considered the coordination principles "In" and "Out," as principles relating assertions and denials. In, for instance, tells us that if one is committed to asserting A, then denying A is incoherent, whereas Out tells us that if it's incoherent to assert A, then one is committed to denying A. Once again, Bilateral LP rejects In but accepts Out, Bilateral K3 rejects Out but accepts In, and Bilateral FDE rejects both. *All* logics, however, can accept the following versions of In and Out relating the making and challenging of moves. For lack of a better term, I'll refer to such principles as *pragmatic* In and Out. Consider first pragmatic In (where φ is an assertion or denial):

$$\frac{\Gamma \vdash \checkmark\varphi}{\Gamma, \text{✗}\varphi \vdash}\ \text{p-In}_1 \qquad\qquad \frac{\Gamma \vdash \text{✗}\varphi}{\Gamma, \checkmark\varphi \vdash}\ \text{p-In}$$

These principles say that if Γ commits one to making some move φ (be it an assertion or denial), then Γ along with challenging φ is incoherent. Likewise, if Γ commits one to challenging φ, then Γ along with making the move φ is incoherent. Clearly, this seems right. Insofar as a challenge to a move functions to undermine the entitlement one has to that move, if one is committed to some move, then it is clearly incoherent to challenge that move, and if one is committed to challenging some move, then it is clearly incoherent to make a move. Now consider the pragmatic Out rules:

$$\frac{\Gamma, \checkmark\varphi \vdash}{\Gamma \vdash \text{✗}\varphi}\ \text{p-Out} \qquad\qquad \frac{\Gamma, \text{✗}\varphi \vdash}{\Gamma \vdash \checkmark\varphi}\ \text{p-Out}$$

These principles say that if Γ along with making the move φ is incoherent, then Γ commits one to challenging φ. Likewise, if Γ along with challenging φ is incoherent, then Γ commits one to making the move φ.[15]

[37] (though with a bilateralist twist), Incurvati and Schlöder understand these acts in terms of their potential to change the common ground, understood as partitioned into the *positive* common ground and the *negative* common ground. The aim of asserting a sentence is to include it in the positive common ground, the aim of denying a sentence is to include it in the negative common ground. The aim of challenging the assertion of a sentence ("weakly denying" it) is to exclude it from the positive common ground, and the aim of challenging the denial of a sentence ("weakly asserting" it) is to exclude it from the negative common ground. The key difference is that Incurvati and Schlöder suppose that there can be no overlap in sentences included in the positive and negative common ground [16, 60-70], whereas the framework here permits such overlaps. As a result, the use of their terminology would be misleading in this context, since, as we'll see shortly, in Bilateral LP, "weak denial" is in fact strictly stronger than "strong denial." I thus opt to speak simply in terms of making and challenging assertions and denials, finding it more perspicuous to deploy a notation that reflects this terminological choice.

[15]The latter of these two principles might seem controversial, If opposing φ is *incoherent*, then φ cannot possibly be

It is indeed trivial to expand the notion of correctness to apply to formulas of the form $\text{✗}\varphi$ in such a way that validates p-In and p-Out: $\text{✗}\varphi$ is correct, relative to a valuation v, just in case φ is incorrect, relative to v.[16] We can show, now that, in Bilateral K3 denying some sentence commits one to challenging the assertion of that sentence, but not vice versa, whereas, in Bilateral LP, challenging the assertion of some sentence commits one to denying that sentence, but not vice versa. For the positive part of this claim, just consider the following two proofs:

$$\dfrac{\dfrac{\dfrac{}{\checkmark{-}A \vdash \checkmark{-}A}\ \text{Reflex.}}{\checkmark{-}A, \checkmark{+}A \vdash}\ \text{In}}{\checkmark{-}A \vdash \text{✗}{+}A}\ \text{p-Out} \qquad\qquad \dfrac{\dfrac{\dfrac{}{\checkmark{+}A \vdash \checkmark{+}A}\ \text{Reflex.}}{\checkmark{+}A, \text{✗}{+}A \vdash}\ \text{p-In}}{\text{✗}{+}A \vdash \checkmark{-}A}\ \text{Out}$$

To see that, in each case, the converse doesn't hold, just note that, if it did, we could derive Out in BK3 and In in BLP. This formalization thus enables us to distinguish between two senses of the "denial" of A. The first sense of "denial," which we may denote *denial*$_1$, is the sense expressed here by $\checkmark{-}A$, which is inferentially equivalent to (though not identical to) the assertion of $\neg A$. The second sense of "denial," which we may denote *denial*$_2$, is the sense expressed here by $\text{✗}{+}A$, whose performance amounts to challenging the assertion of A. We can note that, in Bilateral K3, denying$_1$ A is strictly stronger than denying$_2$ A in that performing the former act commits one to performing the latter but not vise versa, whereas, in Bilateral LP, the exact converse holds. It seems clear that considering these dual gappy and glutty possibilities, in which assertion and denial are not assumed to be exhaustive or exclusive, makes good sense, and this is only possible insofar as logic itself imposes neither the exhaustivity nor the exclusivity of assertion and denial.

Making this two-dimensional bilateral approach explicit in the notation enables us to respond to a number of other concerns that one might have about this approach. I'll focus on two additional concerns that might seem most pressing.

The first additional concern has to do with the fact that bilateralist conception of assertion and denial I've appealed to here in putting forth these paraconsistent systems, according to which denying a sentence is inferentially equivalent to asserting its negation (and thus, a dialetheist ought to both assert and deny a sentence they take to be true and false), is out of line with the use of "assertion" and "denial" by prominent dialetheists such as Priest [27] [28]. Consider, for instance, what Priest [27] says about the assertions and denials made by gap-theorists and glut-theorists:

> Consider someone who supposes that some sentences are neither true nor false. Let A be a sentence that they take to be of this kind. They will then deny A; but their denial is certainly not to be taken as an assertion of $\neg A$.

[16] Given this way of defining correctness for doubly bilateral formulas, it is equally trivial that to show that the expanded proof system, which adds p-In and p-Out, is sound and complete relative to doubly bilateral consequence, and that the above points about the separability of the negation rules hold for this expanded system.

> [...] Conversely, a dialetheist who has ground for believing that A and $\neg A$ are both true may assert $\neg A$ without thereby denying $\neg A$, (104-105).

On the approach I've laid out here, by contrast, if A is "gappy" one should *not* deny A. Conversely, if A is "glutty," one *should* deny $\neg A$ (as well as asserting it). One might wonder, then, if the notions of "assertion" and "denial" at use in these paraconsistent system are not tracking the use of those notions by actual paraconsistent logician, what reason do we have to use such a system to develop paraconsistent theories.

In fact, however, given the distinction just made, we can show that this framework enables us to define notions of assertion and denial that *precisely* track the way in which Priest uses the notions. Given our distinction between the two senses of "denial," and our formalization of the inferential norms governing "denial" in these respective senses, it is clear that, by "denial," Priest means denial$_2$. That is, he means the act of *opposing* an assertion. Indeed, he explicitly articulates his notion of denial as an act through which one expresses disagreement [28, 291-292]. This disambiguation between senses of "denial" shows that any disagreement about the nature of assertion and denial between Priest and a proponent of the bilateral systems laid out here is merely verbal. There is *a* use of denial such that Priest's statement is perfectly correct: the one expressed here by $✗_{+}A$. It's just that the term "denying A" is principally used here to express the act of taking A to be false, expressed here with $✓_{-}A$, which, as explained above, is inferentially equivalent to but crucially distinct from the act of taking $\neg A$ to be true.

A further as of yet unaddressed issue has to do with the fact that the bilateral systems put forward are *multiple conclusion* sequent calculi. Above, when explicating the significance of the negation rules of the familiar *uni*lateral multiple conclusion sequent calculus, I appealed to Restall's bilateral interpretation of multiple conclusion sequents, according to which $X \vdash Y$ is read as saying that asserting everything in X and denying everything in Y is incoherent or "out of bounds." This interpretation is widely appealed to by proponents of inferentialism using multiple conclusion sequents in formally developing their inferentialist theories.[17] However, in the explicitly bilateral framework put forward here, we have *bilateral* multiple conclusion sequents of the form $\Gamma \vdash \Delta$ where the formulas in Γ and Δ are *themselves* assertions and denials. Since assertions and denials cannot themselves be asserted or denied, Restall's bilateral reading of these multiple conclusion sequents is unavailable. Accordingly, it's still not clear that this his framework is really compatible with an inferentialist theory.

However, though the specific bilateral reading of multiple conclusion sequents proposed by Restall is unavailable in this explicitly bilateral context, the same general sort of "bounds consequence" (Ferguson [11]) conception is nevertheless straightforward available. We can read $\Gamma \vdash \Delta$ as saying that *making* all of the moves in Γ and *challenging* all of the moves in Δ is "out of bounds." Thus, the bilateral system can be

[17] See, for instance, [31], [32], [40], and Brandom and Hlobil.

interpreted essentially in Restall-style bilateralist fashion. The axiom of Reflexivity, for instance, amounts to the thought that making some move (be it an assertion or denial) and challenging that very move is always incoherent. Though the proponent of Bilateral LP does not think that asserting and denying the same sentence must be incoherent, they surely do think that asserting and challenging the same (indeed, that's just Priest's point, stated above). Indeed, beyond just interpreting this sequent calculus bilaterally in this way, we can use the doubly bilateral system just above to make this interpretation explicit in the notation itself, just as we made the Restall's bilateral interpretation of unsigned sequents explicit in the notation. Thus, we have the following translation schema:

> **Translation Schema (Round 2):** To translate an bilateral multiple conclusion sequent of the form $\Gamma \vdash \Delta$ to doubly bilateral sequent of the form $\Theta \vdash$, let $\Theta = \{✓\varphi \mid \varphi \in \Gamma\} \cup \{✗\psi \mid \psi \in \Delta\}$.

Applying this translation schema, one can rewrite the above provided sequent calculus such that it features only solely left-sided sequents, encoding incoherence. Eech such sequent corresponds to an equivalence class of single conclusion sequents. Letting $\theta^\star$ denote the *pragmatic opposite* of θ ($✗\varphi$ if θ is of the form $✓\varphi$ and $✓\varphi$ if θ is of the form $✗\varphi$), given Pragmatic In and Out, such doubly bilateral left-sided sequents of the form $\Theta \vdash$ correspond to an equivalence class of sequents of the form $\{\Theta/\theta\} \vdash \theta^\star$ for all $\theta \in \Theta$. Given these equivalences, we can understand, for instance, the positive disjunction right rule:

$$\frac{\Gamma \vdash +\langle A\rangle, +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \vee B\rangle, \Delta}\ +_{\vee_R}$$

As equivalent to the following rules, which are all equivalent to one another:[18]

$$\frac{\Theta, ✗_+\langle A\rangle, ✗_+\langle B\rangle \vdash}{\Theta \vdash ✓_+\langle A \vee B\rangle}\ +_{\vee_R} \qquad \frac{\Theta, ✗_+\langle A\rangle \vdash ✓_+\langle B\rangle}{\Theta \vdash ✓_+\langle A \vee B\rangle}\ +_{\vee_R} \qquad \frac{\Theta, ✗_+\langle B\rangle \vdash ✓_+\langle A\rangle}{\Theta \vdash ✓_+\langle A \vee B\rangle}\ +_{\vee_R}$$

That is, it tells us that one is committed to asserting $A \vee B$, given one's set of moves made and moves challenged, just in case challenging the assertion of A along with challenging the assertion of B is incoherent, or equivalently, just in case challenging the assertion of one of the disjuncts commits one to making the assertion of the other. In this way, the sequent calculus can be understood as providing inferential semantic clauses for the connectives in terms of making and challenging assertions and denials.

[18]See [34] for a defense of rules of this form in a singly bilateral context.

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