11

The Logic of Paradox

11.1 Introduction

You now know the basics of classical logic. Of course, there's much more that can be said and done within the context of classical logic. For instance, I've stated that the systems put forward here are sound and complete, but we didn't actually go through the proofs of those facts. Moreover, as I mentioned at the end of last chapter, if you develop formal systems that builds in certain principles of mathematical reasoning as axioms, you can show that those systems must be *in*complete. If you take more logic, you'll work through those proofs and more. However, I think it's worth concluding the course with a reminder that logic itself isn't tethered to the classical conception of logic that we've developed here; there exists a vast space of possible logics beyond classical logic, and we can apply all the logical tools we've developed here in investigating them. I want to introduce one logic known as the *Logic of Paradox*, or LP for short.

11.2 Motivating LP

LP is a logic in which sentences can be both true and false. Upon hearing that, you might be inclined to just dismiss LP outright as useless or downright absurd. Accordingly, before getting into the nitty gritty details of how LP works, I want to consider three ways of motivating LP.

Motivation One: Counterpossible Reasoning

Let's start by returning to an argument that we considered at the beginning of this course:

Paris is in France. Paris isn't in France. So I'm seventeen feet tall.

When you first looked at this argument, intuitively, this argument doesn't seem like a very good one. However, we now know, that, in the logical systems we've developed, it's valid. We can show this in two ways. First, by constructing the truth-table:

F	$\neg F$	S
T	F	T
T	F	F
F	T	T
F	T	F

As you can see, there's no row where both premises are true, since, in any row in which first premise is true, the second premise is false, and vice versa. Accordingly, there's no row where both premises are true and the conclusion is false. Thus, the argument is valid. As we've already explained, this is simply a consequence of our definition of validity. An argument is valid just in case it is not possible for the premises to be true and the conclusion to be false. Since it's not possible for two contradictory sentences to be true, any argument with contradictory premises is vacuously valid. We can also show that this argument is valid by proving the conclusion from the premises as follows:

1	F	prem.
2	$\neg F$	prem.
3	| $\neg S$	asm.
4	| F	reit. 1
5	| $\neg F$	reit. 2
6	$\neg\neg S$	$\neg_I$ 3-5
7	S	$\neg_E$ 6

So, it's built into how our logical system works that, from a contradiction, anything follows, this argument is valid.

However, suppose it *were* the case that Paris, somehow, was both in France and not in France. Suppose this not in the sense of supposing that the territory of Paris is somehow disputed, or some that some part of Paris is in France and some part of it is not in France (in the way that Istanbul is partly in Europe and partly not in Europe). No, suppose, *per impossible*, that there really is a contradictory state of affairs in the world, such that it's both the case and not the case that Paris is in France. What would follow in this impossible scenario? Perhaps it would follow that Parisians were both French and not French. But would it really follow that I'm seventeen feet tall? Plausibly not!

Reasoning about what *would* happen were something impossible to be the case is known as *counterpossible reasoning*, and, if we want to reason counterpossibly and maintain that logical consequences still hold even in the contradictory scenarios about which we're reasoning, we need a non-classical logic like LP according to which it's not the case that everything follows from a contradiction. Graham Priest, the main proponent of LP, makes this sort of need vivid by telling a story in which the main character comes across a box, and opens it to find, to their shock, that it is both empty and not empty. His description of the scenario is quite compelling:

> Carefully, I broke the tape and removed the lid. The sunlight streamed through the window into the box, illuminating its contents, or lack of them. For some moments I could do nothing but gaze, mouth agape. At first, I thought that it must be a trick of the light, but more careful inspection certified that it was no illusion. The box was absolutely empty, but also had something in it. [. . .]
>
> One cannot explain to a congenitally blind person what the color red looks like. Similarly, it is impossible to explain what the perception of a contradiction, naked and brazen, is like. Sometimes, when one travels on a train, one arrives at a station at the same time as another train. If the other train moves first, it is possible to experience a strange sensation. One's anesthetic senses say that one is stationary; but gazing out of the window says that one is moving. Phenomenologically, one experiences what stationary movement is like. Looking in the box was something like that: te experience was one of occupied emptiness. But unlike the train, this was no illusion. The box was really empty and occupied at the same time.

The story ensues, various other things happen, and various other things don't happen. The details aren't important. What's important is that the story contains a contradiction: the box is both empty and not empty. Nevertheless, there is some determinate set of events that unfold in the story. The box, at the end of the story, is buried. It's not shot off to the moon. If we applied classical logic to think about the events in the story, however, we wouldn't be able to maintain this. Since there's a contradiction, everything would follow! Priest's point, in telling this story, is not to convince you that there really are contradictions in reality; but just to convince that we can coherently reason about scenarios in which there are such contradictions, even if these are only fictional scenarios. If we can reason coherently about such scenarios, then it's reasonable to want to codify how we *ought* to reason in such scenarios, and, if we want to do that, we need a non-classical logic. The Logic of Paradox is one such logic.

Motivation Two: the Liar Paradox

To motivate LP further, let's look another kind of case. Recall in the exercises for Chapter 4, I left you on the Island of Knights and Knaves, the inhabitants of which are either knights, who only utter truths, or knaves, who only utter falsehoods. But, in problem 4.7e, you encountered one inhabitant, Bob, who said, perhaps puzzlingly, "I'm a knave." You presumably saw that there was something funky going on here. If Bob's a knight, then he said something false, and so he can't be a knight, but must be a knave. But if Bob's a knave, then he's said something true, and so he can't be a knave, but must be a knight. But he can't be a knight! So, the only thing to conclude is that the person who gave you this puzzle, namely me, is a knave! I wasn't being honest when I said that every inhabitant is either a knight, who only utters truths, or a knave, who only utters falsehoods, since Bob can neither be a knight nor a knave without there being a contradiction!

Now, in reality, there's no knights or knaves, and so this paradoxical example is not going to lead anyone to think that there are really are contradictions in reality. However, this paradox is very closely related to what's known as the "liar paradox," which has led some people to that conclusion. The liar paradox most commonly exemplifed the following sentence, which we'll call "the liar sentence."

The Liar Sentence: The liar sentence is false.

Is the liar sentence true or false? Well, if it's true, then what it says is true, but what it says is that it is false, and so, if that's true, then it's false. But if it's false, then, given that what it says is just that it's false, it says something true, and so it's true. So, if it's true, then it's false, and if it's false, then it's true. A paradox! Unlike Bob, the inhabitant of the Island of Knights and Knaves, the liar sentence *exists*. Its right there on this page. It doesn't seem meaningless. It seems to say something. So, is what it says true or false? It seems like, whatever we say, we end up contradicting ourselves.

It is worth going through, in detail, how the taking the existence of the liar sentence at face value causes trouble for the classical systems that we've developed in this course. Suppose we want to supplement SL with an additional unary connective T, where T(X) means "X is true." How should we think about the truth-conditions of a sentence of the form "X is true"? Intuitivly, it seems quite clear that "X is true" is true just in case "X" is true and so we should assign the following truth conditions for T

T-Schema: T(X) is true just in case X is true.

Accordingly, if we want to introduce T into our natural deduction system, it's quite clear what its introduction and elimination rules should be. For an introduction rule, from any sentence X, we can infer T(X), and, for an elimination rule, from T(X), we can infer X. These rules seem quite reasonable. Now, suppose we add the liar sentence into our formal language, supplementing Sentential Logic with this sentence, L, such that L is equivalent to $\neg$T(L). If we want to codify this equivalence in our proof system, we can add two primitive inference rules: from $\neg$T(L), we can infer L, and from L, we can infer $\neg$T(L). All this might seem innocuous, but our natural deduction system, supplement with these rules, now enables us to conclude any sentence P. Here's how:

1	$\neg P$	asm
2	L	asm
3	$\neg T(L)$	L_E, 2
4	$T(L)$	T_I, 2
5	$\neg L$	$\neg_I$ 2-4
6	$T(L)$	asm.
7	L	T_E 6
8	$\neg L$	reit. 5
9	$\neg T(L)$	$\neg_I$ 6-8
10	L	L_I, 9
11	$\neg\neg P$	$\neg_I$ 1-10
12	P	$\neg_E$ 11

It should be clear that, in the same way, we could conclude $\neg P$, Q, and every other sentence.

Clearly, we've gone wrong somewhere. But where? It seems like there's three possibilities:

1. In permitting a truth connective like T.
2. In permitting the sort of self-reference exhibited by the liar sentence.
3. In permitting all of the inferences that classical logic allows us to make.

Let's consider these three possibilities in turn. If we prohibit a truth connective like T, claiming that it is in some way defective, this seems tantamount to rejecting the very concept of truth with which we seem to operate. It just seems clear that our concept of truth is such that from "X is true" we can conclude X, and from X we can conclude "X is true." Insofar as we think that this reasoning is coherent, it seems like we should be able to introduce T into our formal language to formally codify it. Thus, insofar as we want to maintain that our concept of truth is coherent, we can't say that (1) is where we went wrong. Now consider (2). If we prohibit the sort of self-reference exhibited by the liar, then, on the same basis, we should rule out as in some way defective sentences like the following:

> This sentence is six words long.

There doesn't seem anything wrong with this sentence. It seems meaningful, and, moreover, it seems to be clearly true. However, it involves precisely the sort of self-reference that is exhibited by the liar. It seems terribly ad hoc to rule out such sentences on the grounds that, if we permitted them, we'd have sentences like the liar. So, (2) doesn't seem like where we went wrong either. This leaves us with (3). Classical logic works on the assumption that all sentences are either true or false and no sentences are both true and false. The inferences it permits are only valid given this assumption. If the liar sentence really is both true and false, and we want to be able to reason about it, we need to *weaken* classical logic so that we can reason about the liar without concluding that every sentence is true. Rejecting the classical principle that everything follows from a contradiction is one way to go here. The Logic of Paradox enables one to go that way.

Motivation Three: Philosophical Interpretation

A final motivation for LP comes from one of the philosophers whose argument we've used as an example several times throughout this course: the 2$^{\text{nd}}$ century Indian philosopher Nāgārjuna. Nāgārjuna is the founder of the *Madhyamaka* school of Buddhist thought, the core doctrine of which is that of the *emptiness* of all things. In particular, all things are empty of "svabhāva," a Sanskrit term that means "intrinsic nature" or, more literally, "own being." Everything, Nāgārjuna claims, lacks any intrinsic nature. The positive account of emptiness that Nāgārjuna gives is that of "dependent origination." All things are what they are only in dependence upon other things, and so nothing is what it is in virtue of its own being. This clearly seems to be Nāgārjuna's view. However, it takes only a moment's thought tol realize that there is a paradox lurking here. On this view, reality as a whole is such that its constituents lack any intrinsic nature, existing only in dependence upon other things. Is it not right, then, to say that this is the nature of reality? Moreover, insofar as we are speaking of the nature of reality *as a whole*—the nature of the *totality* of things—this nature of reality cannot be *dependent*, since there is nothing else but the totality on which the nature of the totality could possibly dependent. So, this nature of reality—that things in reality lack any intrinsic nature—must be intrinsic. Yet, insofar as this *is* the nature of reality as a whole, *everything* must lack intrinsic nature, *including* reality as a whole. Thus, it seems that reality as a whole must both have and lack an intrinsic nature. That, it seems, is just what Nāgārjuna says: "All things have one nature, that is, no nature."

Now, there are exegetical questions as to how Nāgārjuna's philosophy is to be interpreted. However, at least *one* way to try to make clear sense of the *apparently* contradictory view that he seems to have, and the *apparently* contradictory things that he says, is to take it that his view *really is* contradictory, and the things he says to express that view *really are* contradictory. If the ultimate nature of reality really is contradictory, then, of course, the right things to say about the ultimate nature of reality will be contradictions. Thus, this interpretive line at least offers one way to try to charitably interpret what Nāgārjuna says, taking his apparently contradictory statements at face value, and this interpretive line is only possible insofar as we can make sense of there being contradictions in reality that Nāgārjuna could aptly express with contradictory statements. The Logic of Paradox enables one to do this.

Of course, Nāgārjuna is not the only philosopher for whom this sort of interpretive line might be appealing. Many philosophers throughout the course of philosophy's history have seemed to express contradictory views. The 19th century German philosopher Georg Wilhelm Friedrich Hegel, for instance, says the following about motion:

> Something moves, not because at one moment it is here and another there, but because at one and the same moment it is here and not here, because in this "here," it at once is and is not.

Now, Hegel is a notoriously hard philosopher to interpret, but, once again, at least *one* interpretive line here would be to take Hegel's apparent expression of a contradiction at face value, taking it that his view on motion really is contradictory. Or consider the 20th century French philosopher, Jean Paul Sartre. On Sarte's view, the self can only be characterized as a "nothingness," because any moment you take it to be some determinate thing, it can always transcend that determination, identifying itself in opposition to that determination. What, however, about *that very characterization*? Sartre's view of the self seems to have a contradiction at its heart, and Sartre himself seems to endorse that very contradiction, claiming "I am not what I am." Once again, taking it that he really is endorsing a contradiction is not the *only* interpretative line one might take here, but it seems to be at least *one* interpretive line one could take, and LP enables one to take it.

Historical Note:

The "Logic of Paradox" was first explicitly put forward by Graham Priest in his 1979 paper by that title. While logics tolerant of contradictions had been developed prior to Priest's work, none had been quite this simple and intuitive, and, moreover, no logician had really embraced the idea that there really are contradictions in reality before Priest boldly put the thought forward and rigorously defended its intelligibility. As a bit of terminology, a logic that tolerates contradictions in the sense that it's not the case that a contradiction entails everything is said to be *paraconsistent*. One can endorse a paraconsistent logic without thinking that there really are contradictions in reality. Someone who thinks that is called a *dialetheist*. All dialethists, if they want to logically coherent, must endorse some paraconsistent logic, but not all paraconsistent logicians need to be dialetheists (and most are not).

11.3 Semantics for LP

Having motivated LP, let us go on to officially develop this new logic. As always, a logic consists in a formal language, a semantics, and a deductive system. We'll stick on the sentential level in our development of LP, and so our approach will be exactly analogous to our approach to SL. The vocabulary and grammar of LP is exactly the same as that of SL. So, our vocabulary consists in simple sentences like A, B, P, Q, and so on, and the logical connectives $\neg$, $\wedge$, $\vee$, and $\rightarrow$, where complex formulas can be constructed out of these expressions in the usual way.

Like SL, we're going to provide a semantics for LP using *truth-tables*. Here's where the difference comes in. For SL, it was crucial that every sentence was assigned exactly one of two truth-values: true or false. No sentence could be neither true nor false, and no sentence could be both true and false. For LP, we allow an additional truth-value. In addition to being (just) true or (just) false, a sentence can be *both* true and false. So, rather than having just two truth-values, we have *three:* T for (just) true, F for (just) false, and B for both true and false. So, instead of having 2^n truth-possibilities for n simple sentences, we have 3^n truth-possibilities:

> **Truth-Possibilities for LP:** For n simple sentences, there are 3^n truth-possibilities.

As with SL, we'll provide truth-tables for the connectives that will let us specify the truth-value of any complex sentence, relative to some truth-possiblity.

The truth-table for negation is what you'd expect. We know that if X is false, then $\neg X$ is true, and, if X is true, then $\neg X$ is false. So, if X is both true and false, $\neg X$ is both true and false as well: true because X is false, and false because X is true. We can show this with the following truth-table:

Truth-Table for Negation:

X	$\neg X$
T	F
B	B
F	T

As you can see, this simply extends the truth-table from SL with an assignment of a truth-value to the case in which X is both true and false, and the explanation for why we assign the truth-value we do to $\neg X$ in this case follows directly from our way of thinking about negation from SL.

We can reason this way to arrive at the truth-tables for all of the other connectives. As we know from SL $X \wedge Y$ is true if X is true and Y is true, and it's false if X is false or Y is false. So, if X is (just) true and Y is (just) true, then, as before $X \wedge Y$ is (just) true. However, if, for instance, X is (just) true and Y is both true and false, then $X \wedge Y$ will be both true and false, true because X and Y are both true, and false because Y is false. We can show all of the truth-conditions of $X \wedge Y$ with the following truth-table, which is set-up a bit differently than the truth-tables from SL:

Truth-Table for Conjunction:

$\wedge$	T	B	F
T	T	B	F
B	B	B	F
F	F	F	F

You read this table just like you'd read a multiplication table from grade school. So, you can think of the leftmost column as specifying the possible truth-value of X, and the topmost row as specifying the possible truth-values for Y, and everything in the middle—all the cells you reach from going right from some cell in the leftmost column and down from some cell in the topmost row—as specifying the truth-values of $X \wedge Y$. Once again, as you can see, this truth-table simply extends our truth-table from SL.

The same sort of reasoning can be applied to disjunction to yeild the following truth-table:

Truth-Table for Disjunction:

$\vee$	T	B	F
T	T	T	T
B	T	B	B
F	T	B	F

Reason through this table on your own and make sure you understand why the truth-conditions for disjunctions are what they are in LP.

11.4 Validity in LP

In general, a valid argument is one that will never take you from premises you should accept to a conclusion you shouldn't accept. Spelling out this thought in the context of classical logic, we took a valid argument to be one that can never have true premises and a false conclusion. So, in SL, we officially defined such a notion of validity by saying that an argument was valid just in case there is no truth-possibility in which all of the premises are true and the conclusion is false. In LP, however, there can be some false sentences that you should accept, in particular, the sentences that are not *just* false, but *both* true *and* false. Accordingly, to define validity in LP, we need modify the definition of validity from SL as follows:

> **Validity in LP:** An argument is *valid in LP* just in case there is no truth-possibility in which all of the premises are (at least) true and the conclusion is (just) false. That is, there is no row of the truth-table where each premise is assigned T or B and the conclusion is assigned F.

If a sentence is at least true, then you should accept it. If, on the other hand, a sentence is just false, then you shouldn't accept it. Accordingly, this definition of validity preserves the basic thought that an argument is valid just in case it won't take you from premises you should accept to a conclusion you shouldn't accept.

Given this definition of validity, many classical valid arguments are valid in LP. Consider, for instance, that $\neg(P \vee Q) \vDash \neg P$:

$\neg$	(P	$\vee$	Q)	$\neg$	P
F	T	T	T	**F**	T
F	T	T	B	**F**	T
F	T	T	F	**F**	T
F	B	T	T	**B**	B
B	B	B	B	**B**	B
B	B	B	F	**B**	B
F	F	T	T	**T**	F
B	F	B	B	**T**	F
T	F	F	F	**T**	F

As you can see, in every row in which the premise is at least true, the conclusion is at least true. So, this argument is valid. However, certain arguments are not valid. For instance, $P \wedge \neg P$ no longer entails Q. To see this, just consider the row of the truth table in which P is both true and false and Q is just false. In this case, $P \wedge \neg P$ is both true and false, but Q is just false, and so the argument is invalid.

The fact that $P \wedge \neg P$ no longer entails Q is, of course, one of the main desired consequences of our new definition of validity. However, defining validity in the way that we have, other arguments turn out to be invalid which we might not have intended to invalidiate. For instance, disjunctive syllogism—the argument with premises $P \vee Q$ and $\neg P$ and conclusion Q—is also now invalid, as you can see by the following truth-table:

P	$\vee$	Q	$\neg P$	Q
T	**T**	T	**F**	**T**
T	**T**	B	**F**	**B**
T	**T**	F	**F**	**F**
B	**T**	T	**B**	**T**
B	**B**	B	**B**	**B**
B	**B**	F	**B**	**F**
F	**T**	T	**T**	**T**
F	**B**	B	**T**	**B**
F	**F**	F	**T**	**F**

The one truth-possibility that invalidates disjunctive syllogism is the case in which P is both true and false and Q is just false. Since $P \vee Q$ is true just in case one of the disjuncts is true, and P is true, $P \vee Q$ is true. Since P is false, $\neg P$ is true. But, in this case, Q is just false. So, though we should accept both of the premises, since they're both (at least) true, we shouldn't accept the conclusion, since it's (just) false.

This result might feel like an unwelcome one. Disjunctive syllogism seems like a very fundamental inference involving disjunction and negation. Indeed, it was one of our basic natural deduction rules! However, if you think about it a bit, it becomes clear that the only reason disjunctive syllogism orginally seemed so plausible as a valid form of inference was that we were working on the assumption that no sentence could be both true or false, and so, if you know that a sentence is false, you can rule out its truth. So, we reasoned that, if $P \vee Q$ is true, then at least one of P or Q must be true, and if $\neg P$ is true, then that means P is false, and so, given that either P or Q is true, we concluded that Q must be true. But if P could be *both* true and false, then we can't use the fact that P is false to rule out that P is true and conclude from $P \vee Q$ that Q must be true. Thus, disjunctive syllogism is valid only insofar as no sentence can be both true and false.

Still, at this point, you might think that disjunctive syllogism is such a fundamental form of inference that any logic that invalidates it is simply too weak to be of any use. Many philosophers and logicians have drawn this conclusion, disregarding non-classical logics like LP on this basis. Still, I think it's worth emphasizing the specific use cases we outlined at the beginning of this chapter. They all involved very abnormal paradoxical circumstnaces, and it's reasonable to suppose that, when reasoning about such paradoxical circumstances, we need to be much more careful in the inferences that we draw than when we reason in almost every other circumstance, which isn't paradoxical. So, even if you use LP for these cases, it's still obviously reasonable to use the classical systems we developed in this course, with all of the inferences that they permit, for every case that you know *isn't* paradoxical.

11.5 Natural Deduction for LP

We've now gone through the semantics for LP. Just as with SL, we'll have a deductive system for LP that is sound and complete with respect to this semantics. We'll use the same style of natural deduction that we've used throughout this book. However, the rules will be somewhat different. We'll just provide rules for conjunction, disjunction, and negation, leaving the conditional $X \rightarrow Y$ as defined to be equivalent to $\neg(X \wedge \neg Y)$. Let's go through the rules. The conjunction rules are the same:

Conjunction Rules:

j	X		j	$X \wedge Y$		j	$X \wedge Y$	
	$\vdots$			$\vdots$			$\vdots$	
k	Y		k	X	$\wedge_E$ j	k	Y	$\wedge_E$ j
	$\vdots$							
l	$X \wedge Y$	$\wedge_I$ j, k						

The disjunction introduction rules are the same as well:

Disjunction Introduction Rules:

j	X			j	Y		
	$\vdots$				$\vdots$		
k	$X \vee Y$	$\vee_I\ j$		k	$X \vee Y$	$\vee_I\ j$	

Howevever, we can't use our same disjunction elimination rules. As we've already seen, these rules constitute inferences that are invalid in LP. Moreover, as we saw in doing the tricks from Chapter 6, these rules let us conclude anything from a contradiction as follows:

1	X	
2	$\neg X$	
3	$X \vee Y$	$\vee_I$ 1
4	Y	$\vee_E$ 2, 3

Given that we no longer want to be able to infer this way, we need a new elmination rule. The new rule is one you've already seen in the exercises for Chapter 6. It is *proof by cases:*

New Disjunction Elimination Rule:

j	$X \vee Y$	
	$\vdots$	
k	$\quad X$	asm.
	$\quad \vdots$	
l	$\quad Z$	
m	$\quad Y$	asm.
	$\quad \vdots$	
n	$\quad Z$	
o	Z	$\vee_E$ j, k-l, m-n

So, if you can prove Z from X and you can prove Z from Y, then you can prove Z from $X \vee Y$.

What should the negation rules be? We can't have our normal negation rules, since, as we've already seen, they suffice to derrive any formula Y from any contradiction of the form X and $\neg X$. In general, we can't have a reductio rule, since it would enable us to derrive anything from a contradiction by assuming its opposite and reiterating the contradiction into our subproof. Thus, our new negation rules will, like our previous negation elimination rule, both be *double* negation rule:

New Negation Rules:

j	X	
	$\vdots$	
k	$\neg\neg X$	$\neg_I$ j

j	$\neg\neg X$	
	$\vdots$	
k	X	$\neg_E$ j

So, we can move freely between any sentence X and its double negation.

The natural deduction system we've introduced thus far are *sound* with respect to the semantics of LP. However, it's not yet *complete*. For instance, with these rules, we cannot yet prove $\neg P$ from $\neg(P \vee Q)$, which, as we've seen, is valid in LP. In our previous system, we would have proven this as follows:

1	$\neg(P \vee Q)$	prem.
2	$\quad\vert\ P$	asm.
3	$\quad\vert\ P \vee Q$	$\vee_I$ 2
4	$\quad\vert\ \neg(P \vee Q)$	reit. 1
5	$\neg P$	$\neg_I$ 2-4

But this proof uses the negation introduction rule that we no longer have. Accordingly, we need to introduce additional primitive rules for *negated* disjunctions and *negated* conjunctions. Since disjunction and conjunction are *duals*, the rules for negated disjunctions have exactly the same form as the rules for (non-negated) conjunctions and the rules for negated conjunctions have exactly the same form as the rules for non-negated disjunctions. So, the negated disjunction rules are the following:

Negated Disjunction Rules:

j	$\neg X$	
	$\vdots$	
k	$\neg Y$	
	$\vdots$	
l	$\neg(X \vee Y)$	$\neg\vee_I\ j, k$

j	$\neg(X \vee Y)$	
	$\vdots$	
k	$\neg X$	$\neg\vee_E\ k$

j	$\neg(X \vee Y)$	
	$\vdots$	
k	Y	$\neg\vee_E\ k$

and the negated conjunction rules are the following:

Negated Conjunction Rules:

j	$\neg X$	
	$\vdots$	
k	$\neg(X \wedge Y)$	$\neg\wedge_I$

j	$\neg Y$	
	$\vdots$	
k	$\neg(X \wedge Y)$	$\neg\vee_I$

j	$\neg(X \wedge Y)$	
	$\vdots$	
k	$\mid$ $\neg X$	asm.
	$\mid$ $\vdots$	
l	$\mid$ Z	
m	$\mid$ $\neg Y$	asm.
	$\mid$ $\vdots$	
n	$\mid$ Z	
o	Z	$\neg\wedge_E$ *j*, *k*-*l*, *m*-*n*

With these rules, we can prove all of De Morgan's laws. For instance, the proof of $\neg X \wedge \neg Y$ from $\neg(X \vee Y)$ goes as follows:

1	$\neg(X \vee Y)$	
2	$\neg X$	$\neg\vee_E$ 1
3	$\neg Y$	$\neg\vee_E$ 1
4	$\neg X \wedge \neg Y$	$\wedge_I$ 2 3

and the proof of $\neg(X \wedge Y)$ from $\neg X \vee \neg Y$ goes as follows:

1	$\neg X \vee \neg Y$	
2	$\mid$ $\neg X$	asm.
3	$\mid$ $\neg(X \wedge Y)$	$\neg\wedge_I$ 2
4	$\mid$ $\neg Y$	asm.
5	$\mid$ $\neg X \wedge Y$	$\neg\wedge_I$ 4
6	$\neg(X \wedge Y)$	$\vee_E$ 2-5

Still, the system we have so far is not yet complete for LP (it's actually a sound and complete system for a weaker logic called "FDE"). In order to be able to prove everything that is valid in LP, we'll add the following rule:

Excluded Middle:

j	$\quad X$	
	$\quad \vdots$	
k	$\quad Y$	
l	$\quad \neg X$	
	$\quad \vdots$	
m	$\quad Y$	
n	Y	excluded middle *j-k, l-m*

This says that if we can prove some sentence Y from both X and its negation $\neg X$, then we can conclude Y.

It's worth noting explicitly that, because LP is strictly weaker than the classical systems we've developed in this book, all of these rules are *admissible* in SL and PL. That is, you can use any of the rules for LP and you won't be able to prove anything that is invalid in SL or PL. Accordingly, if you're doing an SL or PL proof and it seems convenient to use one of the rules from LP, you can without any fear that you're going to conclude something invalid. The converse, however, doesn't hold. That is, you can't use all of the rules of SL if you want to prove something in LP because, as we've, not all of these rules are valid in LP.

11.6 True Contradictions, but So What?

Let's return to consider the liar sentence L. L, recall, was introduced as equivalent to $\neg T(L)$. What could the truth-value of L be such that this equivalence holds. Well, we can consider all of the possibilities and see that there's only one:

L	$\mathrm{T}(L)$	$\neg\mathrm{T}(L)$
T	T	F
B	B	B
F	F	T

Thus, the only truth-value we can assign to L insofar as it says what it purports to say is both true and false. So, taking L at face value, we can conclude that L is both true and false in every truth-possibility. And, indeed, with the T rules introduced earlier (we've now added negated T rules) and with the L rules, we can prove that L is both true and false. That is, we can prove $L \wedge \neg L$:

1	$\quad L$	asm.
2	$\quad \neg\mathrm{T}(L)$	L_E 1
3	$\quad \neg L$	$\neg\mathrm{T}_E$ 2
4	$\quad L \wedge \neg L$	$\wedge_I$ 1, 2
5	$\quad \neg L$	asm.
6	$\quad \neg\mathrm{T}(L)$	$\neg\mathrm{T}_I$ 5
7	$\quad L$	L_I 6
8	$\quad L \wedge \neg L$	$\wedge_I$ 5, 7
9	$L \wedge \neg L$	excluded middle 1-8

So, $L \wedge \neg L$ is true, which means, of course, that L is both true and false. Note, however, that we can also derrive $\neg(L \wedge \neg L)$:

1	$\quad L \wedge \neg L$	asm.
2	$\quad \neg L$	$\wedge_E$ 1
3	$\quad \neg(L \wedge \neg L)$	$\neg\wedge_I$ 2
4	$\quad \neg(L \wedge \neg L)$	asm.
5	$\quad \neg(L \wedge \neg L)$	reit 4
6	$\neg(L \wedge \neg L)$	excluded middle 1-5

So, not only can we conclude $L\wedge\neg L$, but we can also conclude $(L\wedge\neg L)\wedge\neg(L\wedge\neg L)$. That is, not only is L both true and false, but the very sentence that states this contradiction, $L \wedge \neg L$, is itself both true and false, and, as you might have guessed (and as you can confirm for yourself), the sentence that states *that* contradiction is both true and false, and so on.

So, having introduced the liar into LP, we've introduced an infinite number of true (and false) contradictions. But *so what?* The way we've constructed the logic, nothing bad happens. It doesn't follow that I'm seventeen feet tall or that the moon is made of cheese. The only things that *do* follow are the things that *should* follow, given how we're thinking about contradictions. Thus, though the system we've introduced is *inconsistent* in the sense of containing contradictions, it's perfectly logically *coherent*.

11.7 Conclusion

In the opening chapter of this book, we described logic as "the science of good reasoning." We've now seen that what "good reasoning" is may depend on what sorts of circumstances one is reasoning about. In almost all ordinary circumstances, inferring X from $X \vee Y$ and $\neg Y$ is a bit of good reasoning. In paradoxical circumstances, however, it might be a bit of bad reasoning. Moreover, almost all ordinary circumstances, concluding $X \wedge \neg X$ can only be the result of a bit of bad reasoning. In certain paradoxical circumstances, however, this might be just what one should conclude if one is reasoning well. Now, I've picked quite a drastic example of a non-classical logic in order to make this general point, but, regardless of what you think of this particular example, the general point is worth emphasizing.

There is, I think, a widespread conception of logic as, in some way, *constraining* what you can say or think. There are the so-called "laws of logic," things like the Law of Non-Contradiction, and, if you break them, you're speaking nonsense. Indeed, lots of philosophers have argue that it's not even possible to think that a contradiction is true; try as you might, you just can't do it. To this, philosophers like Graham Preist have replied "watch me." Now, though I offered some motivation for LP at the beginning of this chapter just so you would not dismiss it from the outset, it has not been my intention in this chapter to convince you that there really are contradictions in reality. My intention, rather, has been to give you the formal tools so that you *can* think that there really are contradictions in reality, if you really want to, knowing that you're being perfectly logically coherent in doing so. So, if you're arguing with friend of yours, and you maintain that there is some true contradiction (say, the liar sentence, of perhaps even the ulitmate nature of reality), and your friend says "*No!* That's *illogical! Incoherent!* You *can't* think that!" you can now reply "watch me," and you can explain exactly how it is that you're reasoning coherently about the contradiction you take to obtain.

So, far from *constraining* what you can say or think, logic is capable of *opening up new possibilities* for saying or thinking anything under the Sun that you might want to say or think. If there is a way to coherently think about something, you can use logic to see how it can be done. In this way, logic provides you with a set of tools for thinking clearly about anything at all about which you might want to think. I hope the logical tools that you've learned in this course—limited though they are—will prove to be useful additions to your toolkit for thinking, and I hope you go on to acquire more and more logical tools to aid you in all of your future intellectual endevours.