There is a Logical Negation: "Yes," "No," Both, and Neither

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October 27, 2024

Abstract

Jc Beall agues that if FDE is logic proper, then there is no logical negation. This claim is largely based on the fact that, in standard proof systems for FDE, there are no stand-alone negation rules that suffice to capture the behavior of negation. In this paper, I show that by adopting a *bilateral* proof system for FDE, one can maintain that there is a logical negation, it is the very same logical negation that belongs to classical logic, and its basic function is to flip-flop between assertion and denial. I conclude by responding to the objection that it can never be coherent to both assert and deny the very same thing.

Key Words: Negation; Bilateralism; Non-Classical Logic; Proof-Theoretic Semantics; FDE

0 Introduction

In his paper, "There is No Logical Negation: True, False, Both, Neither," Jc Beall argues, first, that there are reasons to adopt the weak subclassical logic FDE as "logic proper," and, second, that if we do, there is no logical negation. I focus here on the second, conditional claim, granting, at least for the sake of argument, the first claim (though I am indeed sympathetic to it). Though not explicitly articulated in this way, Beall's conclusion about the lack of logical negation can be understood as motivated in broadly proof-theoretic terms. In particular, in standard proof systems for FDE, there are no separable negation rules that characterize the inferential behavior of negation, nor (as in the case of LP or K3) are there axiom schemas involving negation as the sole logical operator. In this paper, I show that, by adopting a bilateral approach to FDE, adopting a proof system in which formulas are positively and negatively signed to express assertion and denial, Beall's reasons for thinking that there is no logical negation vanish. There is a logical negation, it's the very same logical negation that belongs to classical logic, and its basic function is to flip-flop between assertion and denial. Though this bilateral perspective on FDE is extremely natural, it has not been hitherto adopted. Thus, I conclude by considering what I take to be the main reason for that, responding to an objection that it can never be coherent to both assert and deny the same thing. I show that, insofar as this is taken be an objection to the bilateral approach I lay out, it is based on an equivocation between two senses of "denial."

1 FDE as "Logic Proper"

Let me start by briefly laying out the motivation for taking FDE to be "logic proper." In a first logic course, one learns the truth and falsity conditions for logical connectives of negation, conjunction, and disjunction.[1] A negation is true just in case the negatum is false, and a negation is false just in case the negatum is true. Likewise, a conjunction is true just in case both conjuncts are true, and a conjunction is false just in case at least one of the conjuncts is false. Dually for disjunction. Officially, where 1 is truth and 0 is falsity, the truth and falsity conditions for the standard logical connectives are given as follows:

$$v(\neg A) = \begin{cases} 1, & \text{if } 0 = v(A) \\ 0, & \text{if } 1 = v(A) \end{cases}$$

$$v(A \wedge B) = \begin{cases} 1, & \text{if } 1 = v(A) \text{ and } 1 = v(B) \\ 0, & \text{if } 0 = v(A) \text{ or } 0 = v(B) \end{cases}$$

$$v(A \vee B) = \begin{cases} 1, & \text{if } 1 = v(A) \text{ or } 1 = v(B) \\ 0, & \text{if } 0 = v(A) \text{ and } 0 = v(B) \end{cases}$$

In the context of classical logic, we assume that truth and falsity are exclusive and exhaustive, such that no sentence can be both true and false and no sentence can be neither true nor false. These assumptions, however, seem to be *substantive* ones, and both of them have been called into question in certain contexts, the most famous of which are those pertaining to paradoxes such as the liar. Regardless of what ultimately wants to say about paradoxes such as the liar, it seems clear that logic enables us to investigate the consequences of all of the things that one *could* say, where, among logically possible options, are ones that reject exclusivity or exhaustivity. Though there may be compelling reasons to accept exclusivity and/or exhaustivity when dealing with paradoxes such as the liar, these aren't strictly speaking *logical* reasons; *logic itself* doesn't force us into such an acceptance.

If one is moved by considerations of the above sort, then one will think that, as far as logic itself is concerned, we can allow that sentences may have one of four possible valuations: just true (or $\{1\}$), just false (or $\{0\}$), both true and false (or $\{1, 0\}$), or neither true nor false (or $\varnothing$). Adopting this more permissive conception of what is logically possible, we can maintain that the semantic clauses for logical connectives are just those stated above; we simply swap the "=" sign (the use of which involves

[1] I ignore the material conditional here, treating it as defined in terms of these connectives.

the assumption of classicality) with the "∈" sign (the use of which does not involve this assumption):

$$v(\neg A) \ni \begin{cases} 1, & \text{if } 0 \in v(A) \\ 0, & \text{if } 1 \in v(A) \end{cases}$$

$$v(A \wedge B) \ni \begin{cases} 1, & \text{if } 1 \in v(A) \text{ and } 1 \in v(B) \\ 0, & \text{if } 0 \in v(A) \text{ or } 0 \in v(B) \end{cases}$$

$$v(A \vee B) \ni \begin{cases} 1, & \text{if } 1 \in v(A) \text{ or } 1 \in v(B) \\ 0, & \text{if } 0 \in v(A) \text{ and } 0 \in v(B) \end{cases}$$

Relaxing things in this way, and defining validity as preservation of truth in all valuations, we obtain the logic known as FDE (first-degree entailment). Excluding $\varnothing$ from the set of admissible valuations, we get LP. Excluding $\{1, 0\}$, we get K3.[2] Excluding both $\varnothing$ and $\{1, 0\}$, we get Classical Logic. Insofar as logical possibility is maximally broad, it is natural to conceive of FDE as telling us what is logically possible, and stronger logics such as LP, K3, and CL as resulting from excluding certain logical possibilities from consideration. While such an exclusion of possibilities is of course justified in many contexts, this justification is not *logical* justification

2 Unilateral Proof Systems for FDE (and LP, K3, and CL)

Having briefly stated the motivating idea for taking FDE to be "logic proper," let me turn to Beall's claim that, on this conception of logic, "there is no logical negation." Before providing the positive argument that there is logical negation, even on FDE, let me first just say why this is a *prima facie* puzzling thing for Beall to say, if we just look at the semantics stated above. A core idea of retaining the semantic clauses stated above seems to be that the logical connectives we have in FDE are just those that we have in CL. After all, they have the very same semantic clauses. It's just that these semantic clauses operate on a broader space of possibilities for the truth-values of sentences. So, it seems natural to say that, just as logical conjunction is just what it is on the classical conception, so too logical negation is just what it is on the classical conception. Why, then, does Beall conclude that there is no logical negation? The answer, I think, has to do not with the *semantics* of FDE stated above, but with the standard *proof systems* for FDE.

In the context of this paper, I will follow Beall [4] [5] and focus my discussion on the *sequent calculus* presentation of FDE and related logics. All of the points I will

[2]FDE is Anderson and Belnap's [1] logic of "First Degree Entailment." LP is Preist's [18] "Logic of Paradox," first proposed by Asenjo [2]. K3 is Kripke's [15] "Logic of Truth" (see also Kremer [14]). For an introductory overview of these logics, see Beall, Glanzberg and Ripley [6], Chapter 5.

be making in what follows can be made in exactly the same way in the context of *natural deduction* systems for these logics, but I will not pursue these points in those terms here.[3] Let us start with the classical sequent calculus[4]

$$\frac{}{X, A \vdash A, Y}\ \text{Reflex.}$$

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg_{\mathrm{L}} \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \neg_{\mathrm{R}}$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \wedge_{\mathrm{L}} \qquad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\ \wedge_{\mathrm{R}}$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y}\ \vee_{\mathrm{L}} \qquad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\ \vee_{\mathrm{R}}$$

The negation rules, in particular, are part of what give the classical sequent calculus a formal elegance that is not possessed by its natural deduction counterpart. Whereas standard natural deduction systems for classical logic famously lack harmonious negation rules, in the sequent calculus, classical negation is codified by above rules enabling one to "flip-flop" a sentence across the turnstile. It is easy to see how having these rules amounts to imposing Explosion and Excluded Middle. From the axiom of Reflexivity, we have $A \vdash A, B$, and so the left rule enables us to derive $A, \neg A \vdash B$ for any sentence B. Likewise, from Reflexivity, the right rule enables one to derive $\vdash A, \neg A$. Thus, if one is putting forward a sequent calculus for FDE, which enables one to derive neither explosion nor Excluded Middle, one must reject both such negation rules.

Beall [3] puts forward the following sequent calculus for FDE, based on Priest's [20] tableau system:

$$\frac{}{X, A \vdash A, Y}\ \text{Reflex.}$$

$$\frac{X \vdash A, Y}{X \vdash \neg\neg A, Y}\ \neg\neg_{R} \qquad \frac{X, A \vdash Y}{X, \neg\neg A \vdash Y}\ \neg\neg_{L}$$

$$\frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\ \wedge_{R} \qquad \frac{X \vdash \neg A, \neg B, Y}{X \vdash \neg(A \wedge B), Y}\ \neg\wedge_{R}$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \wedge_{L} \qquad \frac{X, \neg A \vdash Y \quad X, \neg B \vdash Y}{X, \neg(A \wedge B) \vdash Y}\ \neg\wedge_{L}$$

[3] See [25] for bilateral natural deduction systems for the FDE family.

[4] This, in particular, is the version of the classical sequent calculus proposed by [13], which has many nice proof-theoretic properties. See [17] for an overview.

$$\frac{X \vdash A, B}{X \vdash A \vee B} \vee_R \qquad \frac{X \vdash \neg A, Y \quad X \vdash \neg B, Y}{X \vdash \neg(A \vee B), Y} \neg\vee_R$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y} \vee_L \qquad \frac{X, \neg A, \neg B \vdash Y}{X, \neg(A \vee B) \vdash Y} \neg\vee_L$$

Notably, this sequent calculus features not only the standard conjunction and disjunction rules, familiar from the classical sequent calculus, but also rules for *negated* conjunctions and disjunctions. For LP, one adds (the multiple conclusion generalization of) Excluded Middle, $X \vdash A, \neg A, Y$ and, for K3, one adds (the multiple conclusion generalization) Explosion, $X, A, \neg A \vdash Y$. Adding both, we get classical logic. Equivalently, for LP one can add classical logic's right negation rule, and, for K3, one can add classical logic's left negation rule.[5] Adding both, of course, gives us classical logic, and, if we do have both then we can get rid of the negated conjunction and disjunction rules, giving us the familiar classical sequent calculus.

If we look at this proof system, it seems that there is a fundamental difference in the treatment of conjunction and disjunction, on the one hand, and the treatment of negation, on the other. On the one hand, it contains the classical rules for conjunction and disjunction showed above. On the other hand, it *can't* contain the classical rules for negation without collapsing into the classical sequent calculus. Accordingly the negation rules must be given in a different way, and, crucially, unlike the classical sequent calculus there are no separable rules for negation that suffice to characterize the distinctive behavior of negation. There are, of course, *double* negation rules, but these rules only suffice to tell us that negation is an involution, but that, of course, is true as well of the operator Beall calls "logical nullation," expressed in English by "It's true that." Thus, in order to classify the inferential behavior of negation, this sequent calculus relies on rules that characterize negation's interaction with the other logical connectives. Now, in LP and K3 there are at least *some* rules that characterize the stand-alone behavior of negation: the axioms of Excluded Middle in LP and Explosion in K3, or, equivalently, $\neg_R$ in LP and $\neg_L$ in K3. In FDE, however, there are *no* such rules. Thus Beall [5], taking FDE to be logic proper, concludes that "there is no logical negation," (15).

3 Bilateralist Negation

What is the negation operator that, supposedly, the classical logician has but the proponent of FDE does not? Whatever it is, it must be characterized by the standard sequent rules of the classical sequent calculus. Once again, they are the following:

[5] Proof is straightforward. Consider just the case of LP. We know adding Excluded Middle yields LP, so for completeness, just note that Excluded Middle is immediately derivable from Reflexivity and $\neg R$. For soundness, it is straightforward to show the admissibility of $\neg R$ in the system for LP with Excluded Middle by induction on proof height.

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg_{\mathrm{L}} \qquad\qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \neg_{\mathrm{R}}$$

But what do these rules actually *say*? In the context of *logical inferentialist* semantic program [9] [7] [28], which takes seriously the idea that the meaning of a logical connective is given by the inferential rules governing its use as codified by a formal proof system, one cannot simply appeal to the validity of these rules relative to classical semantics to justify them. Rather, they must be straightforwardly intelligible as formally codifying inferential norms. In this context, it has been argued that there is a fundamental issue with appealing to multiple conclusion sequent systems: multiple conclusion "arguments," where the premises are collected conjunctively and the conclusions are collected disjunctively, don't seem to correspond to anything in our ordinary inferential practices.[6] In response to this sort of concern, Restall [21] proposes a reading of multiple conclusion sequents, according to which $X \vdash Y$ is understood as saying that *asserting* everything in X along with *denying* everything in Y is incoherent or "out of bounds." Thus, the turnstile is not, in the first instance, playing the role of separating *premises* from *conclusions*, but, rather, of separating *assertions* from *denials*. Reading sequents in this fashion, the left rule says that if, relative to any position consisting in asserting everything in X and denying everything in Y, denying A is out of bounds, then, relative to that position, asserting $\neg A$ is out of bounds. Likewise, the right rule says that if, relative to any position, asserting A is out of bounds, then, relative to that position, denying $\neg A$ is out of bounds. Thus, on this conception, negation *is* a flip-flop operator, but what it's really flipping and flopping between is assertion and denial.

This basic "bilateralist" conception of the function of classical negation as flipping between assertion and denial has been defended in a different formal context by Rumfitt [24], drawing on prior work from Smiley [26]. Rather than using bilateralism to *interpret* Gentzen's multiple conclusion sequent calculus, Rumfitt introduces signs "+" and "−" for assertion and denial to *bilateralize* Gentzen's natural deduction system for classical logic so as to be able to provide harmonious rules for negation. The rules of Rumfitt's system (formulated in "logistic" notation) are the following:

$$\frac{\Gamma \vdash -\langle A \rangle}{\overline{\Gamma \vdash +\langle \neg A \rangle}}\ +_{\neg} \qquad\qquad \frac{\Gamma \vdash +\langle A \rangle}{\overline{\Gamma \vdash -\langle \neg A \rangle}}\ -_{\neg}$$

Reading the turnstile as expressing a relation of committive consequence, these rules can be understood as saying that one is committed to asserting $\neg A$ just in case one is committed to denying A and one is committed to denying $\neg A$ just in case one is committed to asserting A.

These two bilateral conceptions of classical negation are obviously quite close, and it is natural to wonder about the relation between them. In fact, the two conceptions can be formally brought together by transposing the multiple conclusion

[6] See [27] for a sustained statement of this problem.

sequent calculus, as interpreted by Restall, into the sort of signed notation proposed by Rumfitt. The basic idea is this: in a unilateral sequent calculus, a formula of the form $X \vdash$ can be understood as expressing that all of the sentences in X are jointly inconsistent. In a *bilateral* sequent calculus, then, we might take a formula of the form $\Gamma \vdash$, where Γ is a set of signed formulas, to express the same thing: that the set of moves in Γ, be they assertions or denials, are incoherent. This suggests the following translation of multiple conclusion unilateral sequents, on Restall's interpretation, into solely left-sided bilateral sequents, and vice versa:

> **Translation Schema:** To translate an unsigned multiple conclusion sequent of the form $X \vdash Y$ to a signed sequent of the form $\Gamma \vdash$, let $\Gamma = \{+\langle A\rangle \mid A \in X\} \cup \{-\langle B\rangle \mid B \in Y\}$. Conversely, to translate a signed sequent of the form $\Gamma \vdash$ to an unsigned multiple conclusion sequent of the form $X \vdash Y$, let $X = \{A \mid +\langle A\rangle \in \Gamma\}$ and $Y = \{B \mid -\langle B\rangle \in \Gamma\}$.

Translating multiple conclusion sequents in this way, the classical negation rules come out as follows:

$$\frac{\Gamma, -\langle A\rangle \vdash}{\Gamma, +\langle \neg A\rangle \vdash}\ +_{\neg} \qquad\qquad \frac{\Gamma, +\langle A\rangle \vdash}{\Gamma, -\langle \neg A\rangle \vdash}\ -_{\neg}$$

Translated in this way, these two bilateralist conceptions of negation (Rumfitt's, understood in terms of committive consequence, and Restall's, understood in terms of normative incoherence) collapse into one just in case we impose certain *coordination principles*, bilateral structural rules which "coordinate" the opposite speech acts of assertion and denial. In particular, where φ is signed formula (expressing the assertion or denial of some sentence) and φ^* is the oppositely signed formula (expressing the denial or assertion of that sentence), the coordination principles that collapse the two negation rules into one might be most perspicuously states as follows:

$$\frac{\Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash}\ \text{In} \qquad\qquad \frac{\Gamma, \varphi \vdash}{\Gamma \vdash \varphi^*}\ \text{Out}$$

In says that if Γ *commits* one to φ, then Γ along with φ^* is *incoherent*, whereas Out says that if Γ along with φ is *incoherent*, then Γ *commits* one to φ^*. If we *don't* impose such coordination principles, then we need both pairs of negation rules.

Generalizing, we might think of the multiple conclusion negation rules, understood in Restall-style bilateralist fashion, as specifying a particular case of the *premisory* role of asserting or denying a negation (the case in which there is a null set of conclusions), whereas Rumfitt's bilateral rule specify the *conclusory* role of asserting or denying a negation. Thus, we have the following set of rules:

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta}{\Gamma, +\langle \neg A\rangle \vdash \Delta} +_{\neg_L} \qquad \frac{\Gamma, +\langle A\rangle \vdash \Delta}{\Gamma, -\langle \neg A\rangle \vdash \Delta} -_{\neg_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta}{\Gamma \vdash +\langle \neg A\rangle, \Delta} +_{\neg_R} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta}{\Gamma \vdash -\langle \neg A\rangle, \Delta} -_{\neg_R}$$

These rules tell us that asserting a negation has the same role, as either a premise or conclusion, as denying the negatum, and denying a negation has the same role as asserting the negatum. If the bilateralist story about negation is right, then these rules inferentially specify the meaning of negation. Now, if one has (the multiple conclusion generalizations of) In and Out, then, in the context of a multiple conclusion bilateral system, half of these rules are redundant; one can take just the left rules or just the right rules (or, indeed, one can just take the familiar classical negation rules as positive rules). However, if we are inclined to treat FDE as logic proper for the reasons articulated above, we cannot accept In and Out. Consider, for instance, that, given Reflexivity, we have $+\langle p\rangle \vdash +\langle p\rangle, +\langle q\rangle$. Given In, we can conclude $+\langle p\rangle, -\langle p\rangle \vdash +\langle q\rangle$, an explosion principle which says that asserting and denying some sentence p commits one to asserting an arbitrary sentence q. This should not be accepted by the bilateralist proponent of FDE who accepts gluts, thinking that some sentences are both true and false. Insofar as assertion just is a speech act in which one commits oneself to the truth of a sentence and denial is a speech act in which one commits oneself to the falsity of a sentence, accepting gluts amounts to accepting that some sentence are such that they are both to-be-asserted and to-be-denied. However, asserting and denying some sentence should not commit one to asserting everything. Thus, we must reject In. Analogous gappy reasoning applies to the rejection of Out. Thus, for a proponent of FDE, all four negation rules, codifying negation's role as flip-flopping between assertions and denials as both premises and conclusions, are necessary.

This move to a bilateral setting should come as extremely natural to the sub-classical logician, and, especially the maximally sub-classical proponent of FDE. The crucial thought of FDE in the context of the *semantics* is that we must treat truth and falsity conditions as on a par. We cannot reduce falsity conditions to truth conditions, as one can in the context of classical logic where falsity simply aligns with untruth. The corresponding thought in the context of the bilateral *proof-theory* is that we must treat assertion conditions and denial conditions as on a par. We cannot reduce denial conditions to assertion conditions, as one can in the context of classical logic, where the correctness of denial simply aligns with the incorrectness of assertion. Let me now formulate bilateral proof systems for the FDE family containing these bilateral negation rules.

4 Bilateral Proof Systems for FDE (and LP, K3, and CL)

We start by extending the familiar *unilateral* notion of validity, understood as preservation of *truth*, to a *bilateral* notion of validity, understood as preservation of *correctness*. Officially, the correctness of an assertion or denial is defined as follows:

> **Correctness:** Asserting A is *correct*, relative to some valuation v, just in case $1 \in v(A)$. Denying A is *correct*, relative to v, just in case $0 \in v(A)$.

We now define bilateral validity as follows:

> **Bilateral Validity:** An argument of the form $\Gamma \vdash \Delta$ is *bilaterally valid*, relative to a set of admissible valuations V, $\Gamma \vDash_{B_V} \Delta$, just in case there is no $v \in V$ such that all of the stances in Γ are correct and all of the stances in Δ are incorrect.

In this way, we extend the familiar *unilateral* consequence relations of FDE, LP, K3 to *bilateral* consequence relations. The following sequent calculus is sound and complete relative to the bilateral consequence relation of FDE:

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta}\ \text{Reflex.}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta}{\Gamma, +\langle \neg A\rangle \vdash \Delta}\ +_{\neg_L} \qquad \frac{\Gamma, +\langle A\rangle \vdash \Delta}{\Gamma, -\langle \neg A\rangle \vdash \Delta}\ -_{\neg_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta}{\Gamma \vdash +\langle \neg A\rangle, \Delta}\ +_{\neg_R} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta}{\Gamma \vdash -\langle \neg A\rangle, \Delta}\ -_{\neg_R}$$

$$\frac{\Gamma, +\langle A\rangle, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \wedge B\rangle \vdash \Delta}\ +_{\wedge_L} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta \quad \Gamma \vdash +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \wedge B\rangle, \Delta}\ +_{\wedge_R}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta \quad \Gamma, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \wedge B\rangle \vdash \Delta}\ -_{\wedge_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \wedge B\rangle, \Delta}\ -_{\wedge_R}$$

$$\frac{\Gamma, +\langle A\rangle \vdash \Delta \quad \Gamma, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \vee B\rangle \vdash \Delta}\ +_{\vee_L} \qquad \frac{\Gamma \vdash +\langle A\rangle, +\langle B\rangle}{\Gamma \vdash +\langle A \vee B\rangle}\ +_{\vee_R}$$

$$\frac{\Gamma, -\langle A\rangle, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \vee B\rangle \vdash \Delta}\ -_{\vee_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta \quad \Gamma \vdash -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \vee B\rangle, \Delta}\ -_{\vee_R}$$

For LP, we add the principle of *Bilateral Excluded Middle*, $\Gamma \vdash \varphi, \varphi^*, \Delta$ and, for K3, we add *Bilateral Explosion*, $\Gamma, \varphi, \varphi^* \vdash \Delta$. Adding both, we get classical logic. Equivalently, for LP, we can add (the multiple conclusion generalization of) Out, and, for K3, we can add (the multiple conclusion generalization of) In, and, once again, adding both, we get classical logic.[7] This bilateral proof system is obviously quite close to the unilateral one shown above. However, there are three crucial points about this system that deserve emphasis.

The first crucial point is that all of the rules in this system are *separable*. The inferential behavior of negation is given only by the negation rules, not by rules codifying its interaction with other connectives. Moreover, adding any set of rules to the fragment of the sequent calculus not containing those rules constitutes a conservative extension. Separability is widely taken to be key formal constraint,

[7] It should be easy to see that these claims are true. However, the full proofs are provided in [25].

on a par with harmony, in the context of logical inferentialism.[8] If rules are *not* separable—if, for instance, the rules for conjunction are not separable from the rules for negation—then, if we take seriously the that idea knowing the meaning of a connective is mastering the rules governing its use, it would seem that one could not know the meaning of negation without knowing the meaning of conjunction, nor could one know the meaning of conjunction without knowing the meaning of negation. As I explained above, the lack of separable rules for the negation in standard proof systems for FDE is, I think, the main reason that leads Beall to his conclusion that there is no logical negation. This system, in which there are separable rules that precisely characterize the inferential behavior of negation, completely undercuts that reason.

The second closely related crucial point is that the bilateral principles that, added to the sequent calculus for FDE, yield LP or K3 are *not* negation rules. They are, once again, *coordination principles*, bilateral structural rules that "coordinate" the relation between the speech acts of assertion and denial. Beall's thought, transferred into this setting, is that, insofar as logic is maximally topic-neutral, and, in paradoxical contexts these coordination principles can be called into question, *logic itself* does not impose such coordination. At least for the sake of the present paper, I will grant this thought. But to say this is to say *nothing* about negation, since the crucial bilateralist thought is that coordination principles are *not* negation rules; they are distinctively bilateral *structural* rules. Explosion and Excluded Middle can, of course, be expressed with negation. For instance, we can express Explosion using negation as $+\langle A\rangle, +\langle\neg A\rangle \vdash +\langle B\rangle$. However, to think that, because of this it should be understood, fundamentally, as a principle about *negation* is a mistake. For instance, analogously, just because we can express Explosion as $+\langle A \wedge \neg A\rangle \vdash +\langle B\rangle$ does not mean that it's a principle about *conjunction*. Just as it's a mistake to talk about the distinction between "LP conjunction" and "Classical conjunction" on the basis that LP rejects this principle and Classical Logic accepts it, so too is it a mistake to talk about the distinction between "LP negation" and "Classical negation."

This brings us to the final crucial point, which is that, just as the conjunction rules are the same in each logic, so too, the negation rules are *exactly the same* whether one endorses FDE, LP, K3, or CL. Negation is a logical operator such that $\neg A$ is to be asserted just in case A is to be denied, and $\neg A$ is to be denied just in case A is to be asserted. That is what logical negation does; it toggles between a sentence's *truth*, it's being correct to *assert*, and a sentence's *falsity*, its being correct to *deny*. I contend, then, that there is a logical negation, and it just is the logical operator that does just that.

[8] For a recent discussion of separability in the context of logical inferentialism, see [16].

5 The Coherence of Both Asserting and Denying

Let me conclude by addressing what might seem to be the most glaring objection to the bilateral approach to FDE that I've laid out. If one goes in for this bilateral approach to paraconsistency, then one will say that the liar sentence, assigned a value of $\{1, 0\}$ on all valuations, is such that it is correct to assert and it also is correct to deny. However, it is widely assumed that assertion and denial, though they might not be *exhaustive*, must at least be *exclusive*.[9] That is, though there may be some sentences such that *neither* asserting *nor* denying is correct, there cannot be any sentences such that *both* asserting *and* denying are correct. The reason has to do with the speech acts that assertion and denial are taken to be. These acts, it is widely thought, express opposite *attitudes*: acceptance and rejection, and it is thought by many that the nature of these attitudes is such that one cannot consciously adopt both of them with respect to a single sentence. Warren [29], for instance, says "the dispositions that constitute accepting p and those that constitute rejecting p can't be had by the same person at the same time." In the context of paraconsistent logic in particular, one influential discussion of denial and rejection, which I believe is largely responsible for the impression that the bilateralist approach to paraconsistency I've outlined here is untenable, is owed to Priest [19]. Priest takes assertion to express the attitude of acceptance and denial to express the attitude of rejection, and he takes it that the very nature of these is such that one cannot consciously adopt both of them with respect to a single sentence. Here is Priest on this pair of opposite attitudes:

> To accept something is simply to believe it, to have it in one's "belief box", as it were. To reject something is to refuse to believe it: if it is in one's belief box one takes it out, but whether or not it was in there before, one resolves to keep it out, (103).

Since it is clearly incoherent to both assent to something and refuse to assent to it, to both include it and exclude it from one's belief box, it seems that it must be incoherent to accept and reject the same thing, and, accordingly to assert and deny the same thing. A bilateral approach to paraconsistency, then, which rejects the exclusivity of assertion and denial (maintaining, for instance, that the liar sentence is both correct to assert and correct to deny) seems like a theoretical non-starter.

The appearance that this approach is a theoretical non-starter, however, is illusory. It is the result of a failure to be clear about the distinction two speech acts that can aptly called "denial," and, correspondingly, two attitudes that can aptly be called "rejection." The speech act I am calling "denial" in the context of these bilateral proof systems is an act of explicitly *committing oneself to the falsity* of a sentence. As the rules for negation tell us, such a speech act commits one to asserting its negation. Insofar as this act can be understood as expressing rejection, "rejection," in this sense, is taking something to be false. By contrast, "rejection," in the sense discussed by

[9] For rejections of the exhaustiveness of assertion and denial, see Dummett [8], Restall [22]

Priest, is refusing to take something to be true. This corresponds to a different sense of "denial," understood as an act of explicitly refraining from asserting a sentence, thus explicitly *precluding oneself from being entitled to its truth*. In the current literature that extends *bi*lateralism to *multi*-lateralism, owed largely to the work of Incurvati and Schlöder [12] [10] [11], these two acts (or attitudes), are known respectively as *strong* denial (or rejection) and *weak* denial (or rejection).[10] As we will see shortly, these names are misleading in this context. However, sticking with them for now, the notions of assertion and denial at play in these bilateral systems are *strong* notions. However, in the discussions that have been taken to rule out the bilateralist approach to paraconsistency such as Priest's referenced above, it is typically the *weak* notion of denial that is discussed.

Insofar as both gappy and glutty approaches to paradoxes such as the liar are under consideration, neither of these notions of denial logically entails the other. To see this, just note that the proponent of Bilateral K3 "rejects" the liar in the sense of of refusing to assent to its truth, but they don't "reject" the liar in the sense of assenting to its falsity. On the contrary, they explicitly *refuse* to assent to its falsity. On the other hand, the proponent of Bilateral LP "rejects" the liar in the sense of assenting to its falsity, but they don't "reject" the liar in the sense of refusing to assent to its truth. On the contrary, they explicitly *do* assent to its truth. It is in this context, that we can see that calling the sense of rejection that involves refusal to assent "weak" rejection, as Incurvati and Schlöder do, is misleading. In the non-paraconsistent systems developed by Incurvati and Schlöder, "weak" rejection is indeed, as the name suggests, strictly *weaker* than the "strong" rejection: "strongly" denying something commits one to "weakly" denying it, but not vice versa. However, if we consider glutty approaches to bilateralism, where taking something to be true and taking something to be false need not exclude one another, *"weak" rejection isn't weak.* Insofar as we draw this distinction, one can accept that asserting and denying in *Priest's* sense of "denial" is incoherent while nevertheless maintaining that asserting and denying in the sense of "denial" at issue in these systems is perfectly coherent.

Let me spell out the above thought officially by following Incurvati and Schlöder in introducing two new force markers to express "weak" denial and "weak" assertion (from hereon out, I will continue to use those names, just to have a pair of labels but the point just made above about their misleadingness in this context should be kept in mind).[11] Letting + and − continue express the speech acts of asserting and

[10] A closely related distinction is drawn by Ripley [23] between the *strict* denial and *tolerant* denial. In fact, though I follow Incurvati and Schlöder's terminology and use of force markers, it is technically that distinction between speech acts that corresponds most directly to the distinction I am drawing here.

[11] Though I follow their approach in using these force-markers for these speech acts, it's worth being clear that the logic of weak denial that results from supplementing these bilateral systems with $\ominus$ in the way I propose differs in certain important respects from the logic of weak denial that they develop.

denying, let $\oplus$ and $\ominus$ respectively express the speech acts of explicitly refusing to deny and explicitly refusing to assert. The coordination principles of In and Out we considered above related *strong* assertions and denials. In, for instance, tells us that if one is committed to asserting A, then denying A is incoherent, whereas Out tells us that if it's incoherent to assert A, then one is committed to denying A. Once again, Bilateral LP rejects In but accepts Out, Bilateral K3 rejects Out but accepts In, and Bilateral FDE rejects both. Thus K3 and LP both reject one of *strong-strong* principles of In and Out. *All* logics, however, can accept the strong-weak and weak-strong versions of In and Out. Consider first strong-to-weak In:

$$\frac{\Gamma \vdash +A}{\Gamma, \ominus A \vdash} \text{ In (s-w)} \qquad\qquad \frac{\Gamma \vdash -A}{\Gamma, \oplus A \vdash} \text{ In (s-w)}$$

These principles say that if Γ commits one to asserting A, then Γ along with explicitly refusing to assert A is incoherent, and likewise for denial. Clearly, this seems right. Indeed, it seems in some sense *analytic*, simply unpacking the notion of "commitment" at issue here. Now consider the strong-to-weak Out rules:

$$\frac{\Gamma, +A \vdash}{\Gamma \vdash \ominus A} \text{ Out (s-w)} \qquad\qquad \frac{\Gamma - A \vdash}{\Gamma \vdash \oplus A} \text{ Out (s-w)}$$

These principles say that if Γ along with asserting A is incoherent, the Γ commits one to explicitly refusing to assert to A, and likewise for denial. Once again, this seems clearly right, and, indeed, analytic, simply unpacking the notion of "incoherence" at issue here.

It is indeed trivial to expand the notion of correctness to apply to formulas of the form $\oplus A$ and $\ominus A$ in such a way that validates all weak-to-strong and strong-to-weak principles: $\oplus A$ is correct, relative to a valuation v, just in case $-A$ is incorrect relative to v, and $\ominus A$ is correct just in case $+A$ is incorrect. Note now, that, in Bilateral K3, we can derive $-A \vdash \ominus A$, which says that denying A commits one to explicitly refusing to assert A:

$$\cfrac{\cfrac{\cfrac{}{-A \vdash -A}\text{ Reflex}}{-A, +A \vdash}\text{ In (s-s)}}{-A \vdash \ominus A}\text{ Out (s-w)}$$

In Bilateral LP, on the other hand, we can derive $\ominus A \vdash -A$, which says that explicitly refusing to assert A, resolving to keep the assertion of A out of one's commitment box, commits one to denying A:

$$\cfrac{\cfrac{\cfrac{}{+A \vdash +A}\text{ Reflex}}{+A, \ominus A \vdash}\text{ In (s-w)}}{\ominus A \vdash -A}\text{ Out (s-s)}$$

In both cases, however, the converse doesn't hold. What this means is that, though, in Bilateral K3 "weakly denying" is indeed weaker than actually denying, in Bilateral LP the converse is the case; "weakly denying" is in fact *stronger* than actually denying. So, it indeed follows in Bilateral LP that asserting A and "weakly denying" A (i.e. refusing to assert A) is incoherent. That is, we have $+A, \ominus A \vdash$. But the proponent of Bilateral LP can accept *this* fact and still assert and deny the liar, maintaining that doing so is perfectly coherent.

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