There is a Logical Negation: "Yes," "No," Both, Neither

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Abstract

Jc Beall argues that if FDE is logic proper, then there is no logical negation. This claim is largely based on the fact that, in standard proof systems for FDE, there are no stand-alone negation rules that suffice to capture the behavior of negation. In this paper, I show that by adopting a *bilateral* proof system for FDE, one can maintain that there is a logical negation, it is the very same logical negation that belongs to classical logic, and its basic function is to flip-flop between assertion and denial. After laying out the bilateral proof systems on which this claim is technically based, I develop the conception of assertion and denial on which this claim is philosophically based, responding to a number of objections. I conclude by considering the possibility of a different, so-called "Boolean" negation, in the context of this bilateral framework.

Key Words: Negation; Bilateralism; Non-Classical Logic; Proof-Theoretic Semantics; FDE

0 Introduction

In his paper, "There is No Logical Negation: True, False, Both, Neither," Jc Beall argues, first, that there are reasons to adopt the weak subclassical logic FDE as "logic proper," and, second, that if we do, there is no logical negation. I focus here on the second, conditional claim, granting, at least for the sake of argument, the first claim (though I am indeed sympathetic to it). Though not explicitly articulated in this way, Beall's conclusion about the lack of logical negation can be understood as motivated in broadly proof-theoretic terms. In particular, in standard proof systems for FDE, there are no separable negation rules that characterize the inferential behavior of negation, nor (as in the case of LP or K3) are there axiom schemas involving negation as the sole logical operator. In this paper, I show that, by adopting a bilateral approach to FDE, adopting a proof system in which formulas are positively and negatively signed to express assertions and denials, Beall's reasons for thinking that there is no logical negation vanish. There is a logical negation, it's the very same logical negation that belongs to classical logic, and its basic function is to flip-flop between assertion and denial.

Though this bilateral perspective on FDE is extremely natural, it has not been hitherto adopted. The reason for this, I believe, is that the conception of assertion and denial on which it is based, according to which these two notions are not necessarily incompatible, has seemed to many to be a theoretical non-starter, particularly in the context of an inferentialist approach to logical content. Thus, after laying out this bilateral approach to FDE, I develop the conception of assertion and denial on which it is based in some detail, drawing on the inferentialist framework recently put forward by Hlobil and Brandom [14] as well as the technical work of Blasio, Marcos, and Wansing [9]. The core idea being to distinguish between *two orthogonal dimensions* of bilateralism: *reasons for* and *reasons against*, on the one hand, and *assertion* and *denial*, on the other. Making this conception explicit with a new doubly bilateral system enables me to respond to a number of objections. I conclude by considering the possibility of another negation: so called "Boolean negation," putting a new proof-theoretic take on an old issue about the inexpressibility of such an operator for subclassical approaches to paradox.

1 FDE as "Logic Proper"

Let me start by briefly laying out the motivation for taking FDE to be "logic proper." In a first logic course, one learns the truth and falsity conditions for logical connectives of negation, conjunction, and disjunction.[1] A negation is true just in case the negatum is false, and a negation is false just in case the negatum is true. Likewise, a conjunction is true just in case both conjuncts are true, and a conjunction is false just in case at least one of the conjuncts is false. Dually for disjunction. Officially, where 1 is truth and 0 is falsity, the truth and falsity conditions for the standard logical connectives are given as follows:

$$v(\neg A) = \begin{cases} 1, & \text{if } v(A) = 0 \\ 0, & \text{if } v(A) = 1 \end{cases}$$

$$v(A \wedge B) = \begin{cases} 1, & \text{if } v(A) = 1 \text{ and } v(B) = 1 \\ 0, & \text{if } v(A) = 0 \text{ or } v(B) = 0 \end{cases}$$

$$v(A \vee B) = \begin{cases} 1, & \text{if } v(A) = 1 \text{ or } v(B) = 1 \\ 0, & \text{if } v(A) = 0 \text{ and } v(B) = 0 \end{cases}$$

In the context of classical logic, we assume that truth and falsity are exclusive and exhaustive, such that no sentence can be both true and false and no sentence can be neither true nor false. These assumptions, however, seem to be *substantive* ones, and both of them have been called into question in certain contexts, the most famous

[1] I ignore the material conditional here, treating it as defined in terms of these connectives.

of which are those pertaining to paradoxes such as the liar. Regardless of what one ultimately wants to say about paradoxes such as the liar, it seems clear that logic enables us to investigate the consequences of all of the things that one *could* say, where, among logically possible options, are ones that reject exclusivity or exhaustivity. Though there may be compelling reasons to accept exclusivity and/or exhaustivity when dealing with paradoxes such as the liar, these aren't strictly speaking *logical* reasons; *logic itself* doesn't force us into such an acceptance.

If one is moved by considerations of the above sort, then one will think that, as far as logic itself is concerned, we can allow that sentences may have one of four possible valuations: just true (or $\{1\}$), just false (or $\{0\}$), both true and false (or $\{1, 0\}$), or neither true nor false (or $\varnothing$). Adopting this more permissive conception of what is logically possible, we can maintain that the semantic clauses for logical connectives are just those stated above; we simply swap the "=" sign (the use of which involves the assumption of classicality) with the "$\ni$" sign (the use of which does not involve this assumption):[2]

$$v(\neg A) \ni \begin{cases} 1, & \text{if } v(A) \ni 0 \\ 0, & \text{if } v(A) \ni 1 \end{cases}$$

$$v(A \wedge B) \ni \begin{cases} 1, & \text{if } v(A) \ni 1 \text{ and } v(B) \ni 1 \\ 0, & \text{if } v(A) \ni 0 \text{ or } v(B) \ni 0 \end{cases}$$

$$v(A \vee B) \ni \begin{cases} 1, & \text{if } v(A) \ni 1 \text{ or } v(B) \ni 1 \\ 0, & \text{if } v(A) \ni 0 \text{ and } v(A) \ni B \end{cases}$$

Relaxing things in this way, and defining validity as preservation of truth in all valuations, we obtain the logic known as FDE (first-degree entailment). Excluding $\varnothing$ from the set of admissible valuations, we get LP. Excluding $\{1, 0\}$, we get K3.[3] Excluding both $\varnothing$ and $\{1, 0\}$, we get Classical Logic. Insofar as logical possibility is maximally broad, it is natural to conceive of FDE as telling us what is logically possible, and stronger logics such as LP, K3, and CL as resulting from excluding certain logical possibilities from consideration. While such an exclusion of possibilities is of course justified in many contexts, this justification is not *logical* justification

[2] My use of the backwards "$\in$" sign is a bit non-standard here. I use it just to show how the familiar classical clauses can be directly transformed into 4-valued ones. $A \ni a$, which can be read as "A contains a," is identical to $a \in A$.

[3] FDE is Anderson and Belnap's [1] logic of "First Degree Entailment." LP is Preist's [26] "Logic of Paradox," first proposed by Asenjo [2]. K3 is most famously deployed by Kripke [21] (see also Kremer [19]). For an introductory overview of these logics, see Beall, Glanzberg and Ripley [6], Chapter 5.

2 Unilateral Proof Systems for FDE (and LP, K3, and CL)

Having briefly stated the motivating idea for taking FDE to be "logic proper," let me turn to Beall's claim that, on this conception of logic, "there is no logical negation." Before providing the positive argument that there is logical negation, even on FDE, let me first just say why this is a *prima facie* puzzling thing for Beall to say, if we just look at the semantics stated above. A core idea of retaining the semantic clauses stated above seems to be that the logical connectives we have in FDE are just those that we have in CL. After all, they have the very same semantic clauses. It's just that these semantic clauses operate on a broader space of possibilities for the truth-values of sentences. So, it seems natural to say that, just as logical conjunction is just what it is on the classical conception, so too logical negation is just what it is on the classical conception. Why, then, does Beall conclude that there is no logical negation? The answer, I think, has to do not with the *semantics* of FDE stated above, but with the standard *proof systems* for FDE.

In the context of this paper, I will follow Beall [4] [5] and focus my discussion on the *sequent calculus* presentation of FDE and related logics. All of the points I will be making in what follows can be made in exactly the same way in the context of *natural deduction* systems for these logics, but I will not pursue these points in those terms here.[4] Let us start with the classical sequent calculus[5]

$$\frac{}{X, A \vdash A, Y}\ \text{Reflex.}$$

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg_L \qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \neg_R$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \wedge_L \qquad \frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\ \wedge_R$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y}\ \vee_L \qquad \frac{X \vdash A, B, Y}{X \vdash A \vee B, Y}\ \vee_R$$

The negation rules, in particular, are part of what give the classical sequent calculus a formal elegance that is not possessed by its natural deduction counterpart. Whereas standard natural deduction systems for classical logic famously lack harmonious negation rules, in the sequent calculus, classical negation is codified by above rules enabling one to "flip-flop" a sentence across the turnstile. It is easy to see how having these rules amounts to imposing Explosion and Excluded Middle. From the axiom

[4] See [omitted] for bilateral natural deduction systems for the FDE family. The operational rules of these systems are precisely those of (implication-free fragment of) Rumfitt's [37] classical system that Kürbis [22] calls "$\mathfrak{B}$." As with the sequent calculi put forward in what follows here, all that changes between the systems are the coordination principles (in particular, the versions of Bilateral Explosion and Excluded Middle considered by del Valle-Inclan [11]).

[5] This, in particular, is the version of the classical sequent calculus proposed by [18], which has many nice proof-theoretic properties. See [24] for an overview.

of Reflexivity, we have $A \vdash A, B$, and so the left rule enables us to derive $A, \neg A \vdash B$ for any sentence B. Likewise, from Reflexivity, the right rule enables one to derive $\vdash A, \neg A$. Thus, if one is putting forward a sequent calculus for FDE, which enables one to derive neither explosion nor Excluded Middle, one must reject both such negation rules.

Beall [3] puts forward the following sequent calculus for FDE, based on Priest's [30] tableau system:

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{Reflex.}$$

$$\frac{X \vdash A, Y}{X \vdash \neg\neg A, Y}\ \neg\neg_R \qquad \frac{X, A \vdash Y}{X, \neg\neg A \vdash Y}\ \neg\neg_L$$

$$\frac{X \vdash A, Y \quad X \vdash B, Y}{X \vdash A \wedge B, Y}\ \wedge_R \qquad \frac{X \vdash \neg A, \neg B, Y}{X \vdash \neg(A \wedge B), Y}\ \neg\wedge_R$$

$$\frac{X, A, B \vdash Y}{X, A \wedge B \vdash Y}\ \wedge_L \qquad \frac{X, \neg A \vdash Y \quad X, \neg B \vdash Y}{X, \neg(A \wedge B) \vdash Y}\ \neg\wedge_L$$

$$\frac{X \vdash A, B}{X \vdash A \vee B}\ \vee_R \qquad \frac{X \vdash \neg A, Y \quad X \vdash \neg B, Y}{X \vdash \neg(A \vee B), Y}\ \neg\vee_R$$

$$\frac{X, A \vdash Y \quad X, B \vdash Y}{X, A \vee B \vdash Y}\ \vee_L \qquad \frac{X, \neg A, \neg B \vdash Y}{X, \neg(A \vee B) \vdash Y}\ \neg\vee_L$$

Notably, this sequent calculus features not only the standard conjunction and disjunction rules, familiar from the classical sequent calculus, but also rules for *negated* conjunctions and disjunctions. For LP, one adds (the multiple conclusion generalization of) Excluded Middle, $X \vdash A, \neg A, Y$ and, for K3, one adds (the multiple conclusion generalization) Explosion, $X, A, \neg A \vdash Y$. Adding both, we get classical logic. Equivalently, for LP one can add classical logic's right negation rule, and, for K3, one can add classical logic's left negation rule.[6] Adding both, of course, gives us classical logic, and, if we do have both then we can get rid of the negated conjunction and disjunction rules, giving us the familiar classical sequent calculus.

If we look at this proof system, it seems that there is a fundamental difference in the treatment of conjunction and disjunction, on the one hand, and the treatment of negation, on the other. On the one hand, it contains the classical rules for conjunction and disjunction shown above. On the other hand, it *can't* contain the classical rules for negation without collapsing into the classical sequent calculus. Accordingly the negation rules must be given in a different way, and, crucially, unlike the classical sequent calculus, there are no separable rules for negation that suffice to characterize

[6] Proof is straightforward. Consider just the case of LP. We know adding Excluded Middle yields LP, so for completeness, just note that Excluded Middle is immediately derivable from Reflexivity and $\neg R$. For soundness, it is straightforward to show the admissibility of $\neg R$ in the system for LP with Excluded Middle by induction on proof height.

the distinctive behavior of negation. There are, of course, *double* negation rules, but these rules only suffice to tell us that negation is an involution, but that, of course, is true as well of the operator Beall calls "logical nullation," expressed in English by "It's true that." Thus, in order to classify the inferential behavior of negation, this sequent calculus relies on rules that characterize negation's interaction with the other logical connectives. Now, in LP and K3 there are at least *some* rules that characterize the stand-alone behavior of negation: the axioms of Excluded Middle in LP and Explosion in K3, or, equivalently, $\neg_R$ in LP and $\neg_L$ in K3. In FDE, however, there are *no* such rules. Thus Beall [5], taking FDE to be logic proper, concludes that "there is no logical negation," (15).

Now, one might think that Beall's claim is too strong, but what I'm really concerned to address here is the weaker, conditional claim: *If* you think think that "logical negation" is essentially an operator that's characterized by rules of this sort, *then* there is no negation. What I'll now show is that, given one prominent understanding of what the classical negation rules actually say, the proponent of FDE can accept precisely these rules.

3 Bilateralist Negation

What is the negation operator that, supposedly, the classical logician has but the proponent of FDE does not? Whatever it is, it must be characterized by the standard sequent rules of the classical sequent calculus. Once again, they are the following:

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\ \neg_L \qquad\qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\ \neg_R$$

But what do these rules actually *say*? In the context of the *logical inferentialist* semantic program [13] [10] [45], which takes seriously the idea that the meaning of a logical connective is given by the inferential rules governing its use as codified by a formal proof system, one cannot simply appeal to the validity of these rules relative to classical semantics to justify them. Rather, they must be straightforwardly intelligible as formally codifying inferential norms. In this context, it has been argued that there is a fundamental issue with appealing to multiple conclusion sequent systems: multiple conclusion "arguments," where the premises are collected conjunctively and the conclusions are collected disjunctively, don't seem to correspond to anything in our ordinary inferential practices.[7] In response to this sort of concern, Restall [33] proposes a reading of multiple conclusion sequents, according to which $X \vdash Y$ is understood as saying that *asserting* everything in X along with *denying* everything in Y is incoherent or "out of bounds." Thus, the turnstile is not, in the first instance, playing the role of separating *premises* from *conclusions*, but, rather, of separating *assertions* from *denials*. Reading sequents in this fashion, the left rule says that

[7]See [44] for a sustained statement of this problem.

if, relative to any position consisting in asserting everything in X and denying everything in Y, denying A is out of bounds, then, relative to that position, asserting $\neg A$ is out of bounds. Likewise, the right rule says that if, relative to any position, asserting A is out of bounds, then, relative to that position, denying $\neg A$ is out of bounds. Thus, on this conception, negation *is* a flip-flop operator, but what it's really flipping and flopping between is assertion and denial.

This basic "bilateralist" conception of the function of classical negation as flipping between assertion and denial has been defended in a different formal context by Rumfitt [37], drawing on prior work from Smiley [42].[8] Rather than using bilateralism to *interpret* Gentzen's multiple conclusion sequent calculus, Rumfitt introduces signs "+" and "−" for assertion and denial to *bilateralize* Gentzen's natural deduction system for classical logic so as to be able to provide harmonious rules for negation. The negation introduction rules in Rumfitt's system (formulated in "logistic" notation) are the following:

$$\frac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash +\langle \neg A \rangle}\ +_{\neg} \qquad\qquad \frac{\Gamma \vdash +\langle A \rangle}{\Gamma \vdash -\langle \neg A \rangle}\ -_{\neg}$$

Reading the turnstile as expressing a relation of committive consequence, these rules can be understood as saying that one is committed to asserting $\neg A$ just in case one is committed to denying A and one is committed to denying $\neg A$ just in case one is committed to asserting A.

These two bilateral conceptions of classical negation are obviously quite close, and it is natural to wonder about the relation between them. In fact, the two conceptions can be formally brought together by transposing the multiple conclusion sequent calculus, as interpreted by Restall, into the sort of signed notation proposed by Rumfitt. The basic idea is this: in a unilateral sequent calculus, a formula of the form $X \vdash$ can be understood as expressing that all of the sentences in X are jointly inconsistent. In a *bilateral* sequent calculus, then, we might take a formula of the form $\Gamma \vdash$, where Γ is a set of signed formulas, to express the same thing: that the set of moves in Γ, be they assertions or denials, are incoherent. This suggests the following translation of multiple conclusion unilateral sequents, on Restall's interpretation, into solely left-sided bilateral sequents, and vice versa:

> **Translation Schema:** To translate an unsigned multiple conclusion sequent of the form $X \vdash Y$ to a signed sequent of the form $\Gamma \vdash$, let $\Gamma = \{+\langle A \rangle \mid A \in X\} \cup \{-\langle B \rangle \mid B \in Y\}$. Conversely, to translate a signed sequent of the form $\Gamma \vdash$ to an unsigned multiple conclusion sequent of the form $X \vdash Y$, let $X = \{A \mid +\langle A \rangle \in \Gamma\}$ and $Y = \{B \mid -\langle B \rangle \in \Gamma\}$.

Translating multiple conclusion sequents in this way, the classical negation rules come out as follows:

[8] See also [?] for a notable early development of such an approach.

$$\frac{\Gamma, -\langle A\rangle \vdash}{\Gamma, +\langle \neg A\rangle \vdash}\ +_{\neg} \qquad\qquad \frac{\Gamma, +\langle A\rangle \vdash}{\Gamma, -\langle \neg A\rangle \vdash}\ -_{\neg}$$

Translated in this way, these two bilateralist conceptions of negation (Rumfitt's, understood in terms of committive consequence, and Restall's, understood in terms of normative incoherence) collapse into one just in case we impose certain *coordination principles*, bilateral structural rules which "coordinate" the opposite speech acts of assertion and denial. In particular, where φ is a signed formula (expressing the assertion or denial of some sentence) and φ^* is the oppositely signed formula (expressing the denial or assertion of that sentence), the coordination principles that collapse the two negation rules into one might be most perspicuously stated as follows:

$$\frac{\Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash}\ \text{In} \qquad\qquad \frac{\Gamma, \varphi \vdash}{\Gamma \vdash \varphi^*}\ \text{Out}$$

In says that if Γ *commits* one to φ, then Γ along with φ^* is *incoherent*, whereas Out says that if Γ along with φ is *incoherent*, then Γ *commits* one to φ^*. If we *don't* impose such coordination principles, then we need both pairs of negation rules.

Generalizing, we might think of the multiple conclusion negation rules, understood in Restall-style bilateralist fashion, as specifying a particular case of the *premisory* role of asserting or denying a negation (the case in which there is a null set of conclusions), whereas Rumfitt's bilateral rules specify the *conclusory* role of asserting or denying a negation. Putting these two sets of rules together, then, we have the following set of rules:

$$\frac{\Gamma, -\langle A\rangle \vdash \varphi}{\Gamma, +\langle \neg A\rangle \vdash \varphi}\ +_{\neg_L} \qquad \frac{\Gamma, +\langle A\rangle \vdash \varphi}{\Gamma, -\langle \neg A\rangle \vdash \varphi}\ -_{\neg_L} \qquad \frac{\Gamma \vdash -\langle A\rangle}{\Gamma \vdash +\langle \neg A\rangle}\ +_{\neg_R} \qquad \frac{\Gamma \vdash +\langle A\rangle}{\Gamma \vdash -\langle \neg A\rangle}\ -_{\neg_R}$$

where $\{\varphi\}$ can be null in the left rules

Specifying the inferential role of a logical connective with a sequent calculus containing left and right rules is one way of providing a "two-aspect model of meaning" of the sort associated with Dummett, the left rules specifying the inferential role of a sentence containing that connective *as a premise* and the right rule specifying the inferential role of a sentence containing that connective *as a conclusion*.[9] Together, these rules tell us that asserting a negation has the same role, as either a premise or conclusion, as denying the negatum, and denying a negation has the same role, as either a premise or conclusion, as asserting the negatum. If the bilateralist story about negation is right, then these rules inferentially specify the meaning of negation. Let me now formulate bilateral proof systems for the FDE family containing these bilateral negation rules.

[9]See [20] for an account of how the sequent calculus can be understood as providing an inferentialist theory of logical content in this way.

4 Bilateral Proof Systems for FDE (and LP, K3, and CL)

Standard unilateral proof systems are for the logics in the FDE family are sound and complete with respect to *unilateral* validity. Concretely, the multiple conclusion sequent systems shown above are sound and complete relative to the the following notion of validity:

> **Unilateral Validity:** An argument of the form $X \vdash Y$ is *unilaterally valid*, relative to a set of valuations V, $X \vDash_{U_V} Y$, just in case there's no $v \in V$ such that $1 \in v(A)$ for all $A \in X$, and $1 \notin v(B)$

So *unilateral* validity, at least of this standard variety, is preservation of *truth*. *Bilateral* validity, of the sort appealed to by Smiley [42] and Rumfitt [36] [37], is preservation of *correctness*. Officially, the correctness of an assertion or denial is defined as follows:

> **Correctness:** Asserting A is *correct*, relative to some valuation v, just in case $1 \in v(A)$. Denying A is *correct*, relative to v, just in case $0 \in v(A)$.

Referring to assertions or denials generally as linguistic *moves* one might make, we now define bilateral validity as follows, where Γ and Δ are both sets of signed formulas:

> **Bilateral Validity:** An argument of the form $\Gamma \vdash \Delta$ is *bilaterally valid*, relative to a set of valuations V, $\Gamma \vDash_{B_V} \Delta$, just in case there is no $v \in V$ such that all of the moves in Γ are correct and all of the moves in Δ are incorrect.

In this way, we extend the familiar *unilateral* consequence relations of FDE, LP, K3 to *bilateral* consequence relations. It is easy to see that $+\langle X \rangle \vDash_{B_V} +\langle Y \rangle$ just in case $X \vDash_{U_V} Y$ (where $+\langle X \rangle$ is $\{+A \mid A \in X\}$).

It is possible to formulate both single conclusion and multiple conclusion bilateral sequent calculi for all of the logics in the FDE family containing the negation rules stated in the previous section. The former sort of calculus is more straightforwardly suited to an inferentialist account of meaning, since a single conclusion sequent can be straightforwardly understood as expressing a relation of committive consequence. However, the multiple conclusion sequent calculus is technically nicer, and, given the duality of LP and K3, it is often philosophically illuminating to work in a multiple conclusion setting. Thus, I will introduce both such systems, with the former serving as the main proposal for specifying the content of negation in inferential terms. I take it that the latter can also be taken to provide an inferentialist account of the meanings of the connectives, though it raises the issue of the interpretation of bilateral multiple conclusion sequents, since, given that the formulas in Γ and Δ are singed to express assertions and denials, Restall's bilateral interpretation discussed above is not straightforwardly available. In the end of this paper I will show how the basic bilateralist approach to consequence is in fact still available here, albeit in a different form, and thus, this sequent calculus too can be understood as providing an inferentialist account of the logical connectives. For the moment, however, just think of a bilateral multiple conclusion in terms of correctness preservation, in the sense just defined. Though there remain some issues with doing this in the context of inferentialism (which will be addressed in due time), the most pressing issue for our present purposes—the appeal to the multiple conclusion set-up to provide the negation rules—is gone: we simply have the multiple conclusions generalizations of the bilateral rules for negation shown above.

The key thought regarding both sorts of systems is that all of the operational rules for all of the logics in the FDE family are *exactly the same*, regardless of which logic one is using. All that differs are the *coordination principles*: the bilateral structural rules "coordinating" the relation between assertion and denial. In a single conclusion setting, beyond the principles of In and Out stated above, the two most notable coordiantion principles are (the meta-inferential variants of) Bilateral Explosion and Bilateral Excluded Middle:[10]

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi^*}{\Gamma \vdash \psi}\ \text{(m) B. Explo.} \qquad\qquad \frac{\Gamma, \varphi \vdash \psi \quad \Gamma, \varphi^* \vdash \psi}{\Gamma \vdash \psi}\ \text{(m) B. Ex. Mid.}$$

where $\{\psi\}$ can be null

Clearly, if we are inclined to treat FDE as logic proper, we cannot accept either of these principles. Consider, for instance, that, given Reflexivity, we have $+\langle p\rangle, -\langle p\rangle \vdash +\langle p\rangle$ and $+\langle p\rangle, -\langle p\rangle \vdash -\langle p\rangle$. Given Bilateral Explosion, we can conclude $+\langle p\rangle, -\langle p\rangle \vdash +\langle q\rangle$, an explosion principle which says that asserting and denying some sentence p commits one to asserting an arbitrary sentence q. This should not be accepted by the bilateralist proponent of FDE who accepts gluts, thinking that some sentences are both true and false. Insofar as assertion just is a speech act in which one commits oneself to the truth of a sentence and denial is a speech act in which one commits oneself to the falsity of a sentence, accepting gluts amounts to accepting that some sentence are such that they are both to-be-asserted and to-be-denied. However, asserting and denying some sentence should not commit one to asserting everything. Thus, we must reject B. Explo. Analogous gappy reasoning applies to the rejection of Bilateral Excluded Middle. Switching to a multiple conclusion setting, consider now the multiple conclusion generalizations of the principles I above called "In" and "Out":

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \varphi^* \vdash \Delta}\ \text{In} \qquad\qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \varphi^*, \Delta}\ \text{Out}$$

Similar glutty reasoning leads us to reject In, whereas gappy reasoning leads us to reject Out. Indeed, in the context of structural rules, In and Out are equivalent to the axiom schemas of Bilateral Explosion and Bilateral Excluded Middle:

[10] These are discussed, in the context of bilateral natural deduction systems for classical logic, by del Valle-Inclan [11]. Together

$$\frac{}{\Gamma, \varphi, \varphi^* \vdash \Delta} \text{(a) B. Explo} \qquad \frac{}{\Gamma \vdash \varphi, \varphi^*, \Delta} \text{(a) B. Ex. Mid.}$$

Bilateral Explosion can be understood as saying it's never both correct to assert and correct to deny some sentence, whereas Bilateral Excluded Middle can be understood as saying that it's never both incorrect to assert and incorrect to deny some sentence. Including these axiom shchemas thus directly imposes the exclusivity and exhausitivity of assertion and denial.[11]

Having explained these coordination principles, let me now state the bilateral sequent calculi. The following sequent calculus is sound and complete relative to the single conclusion bilateral consequence relation of FDE:

$$\frac{}{\Gamma, \varphi \vdash \varphi} \text{Reflex.}$$

$$\frac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash +\langle \neg A \rangle} +_{\neg_R} \qquad \frac{\Gamma \vdash +\langle A \rangle}{\Gamma \vdash -\langle \neg A \rangle} -_{\neg_R} \qquad \frac{\Gamma, -\langle A \rangle \vdash \varphi}{\Gamma, +\langle \neg A \rangle \vdash \varphi} +_{\neg_L} \qquad \frac{\Gamma, +\langle A \rangle \vdash \varphi}{\Gamma, -\langle \neg A \rangle \vdash \varphi} -_{\neg_L}$$

$$\frac{\Gamma \vdash +\langle A \rangle \quad \Gamma \vdash +\langle B \rangle}{\Gamma \vdash +\langle A \wedge B \rangle} +_{\wedge_R} \qquad \frac{\Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \wedge B \rangle} -_{\wedge_{R_1}} \qquad \frac{\Gamma \vdash -\langle A \rangle}{\Gamma \vdash -\langle A \wedge B \rangle} -_{\wedge_{R_2}}$$

$$\frac{\Gamma, +\langle A \rangle, +\langle B \rangle \vdash \varphi}{\Gamma, +\langle A \wedge B \rangle \vdash \varphi} +_{\wedge_L} \qquad \frac{\Gamma, -\langle A \rangle \vdash \varphi \quad \Gamma, -\langle B \rangle \vdash \varphi}{\Gamma, -\langle A \wedge B \rangle \vdash \varphi} -_{\wedge_L}$$

$$\frac{\Gamma \vdash +\langle B \rangle}{\Gamma \vdash +\langle A \vee B \rangle} +_{\vee_{R_1}} \qquad \frac{\Gamma) \vdash +\langle A \rangle}{\Gamma \vdash +\langle A \vee B \rangle} +_{\vee_{R_2}} \qquad \frac{\Gamma, +\langle A \rangle \vdash \varphi \quad \Gamma, +\langle B \rangle \vdash \varphi}{\Gamma, +\langle A \vee B \rangle \vdash \varphi} +_{\vee_L}$$

$$\frac{\Gamma, -\langle A \rangle, -\langle B \rangle \vdash \varphi}{\Gamma, -\langle A \vee B \rangle \vdash \varphi} -_{\vee_L} \qquad \frac{\Gamma \vdash -\langle A \rangle \quad \Gamma \vdash -\langle B \rangle}{\Gamma \vdash -\langle A \vee B \rangle} -_{\vee_R}$$

where $\{\varphi\}$ can be null in the left rules

For Bilateral LP, we add (meta-inferential) Bilateral Excluded Middle. For Bilateral K3, we add Bilateral Explosion. Adding both, we get Bilateral CL. The following sequent calculus is sound and complete relative to the multiple conclusion bilateral consequence relation of FDE:[12]

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta} \text{Reflex.}$$

[11] As we'll see, in the context of a bilateral consequence relation, including one of these axioms but excluding the other, will give us the bilateral variants of K3 and LP. It's worth noting, however, that if we instead focus on the *left-side* of these consequence relations, under the translation schema, we see the substructural logics ST and TS, with the bilateral interpretation of them made explicit in the notation. For these results and a discussion of their philosophical consequences, particularly when it comest to the debate between subclassical and substructural approaches to paradox, see [reference ommitted].

[12] This sequent calculus is technically very close to "4-sided" or "4-signed" sequent calculi (e.g. [39], [47]) though conceptually quite different in that there is a single bilateral consequence relation, understood exactly as proposed by Smiley and Rumfitt.

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta}{\Gamma, +\langle \neg A\rangle \vdash \Delta} +_{\neg L} \qquad \frac{\Gamma, +\langle A\rangle \vdash \Delta}{\Gamma, -\langle \neg A\rangle \vdash \Delta} -_{\neg L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta}{\Gamma \vdash +\langle \neg A\rangle, \Delta} +_{\neg R} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta}{\Gamma \vdash -\langle \neg A\rangle, \Delta} -_{\neg R}$$

$$\frac{\Gamma, +\langle A\rangle, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \wedge B\rangle \vdash \Delta} +_{\wedge L} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta \quad \Gamma \vdash +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \wedge B\rangle, \Delta} +_{\wedge R}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta \quad \Gamma, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \wedge B\rangle \vdash \Delta} -_{\wedge L} \qquad \frac{\Gamma \vdash -\langle A\rangle, -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \wedge B\rangle, \Delta} -_{\wedge R}$$

$$\frac{\Gamma, +\langle A\rangle \vdash \Delta \quad \Gamma, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \vee B\rangle \vdash \Delta} +_{\vee L} \qquad \frac{\Gamma \vdash +\langle A\rangle, +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \vee B\rangle, \Delta} +_{\vee R}$$

$$\frac{\Gamma, -\langle A\rangle, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \vee B\rangle \vdash \Delta} -_{\vee L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta \quad \Gamma \vdash -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \vee B\rangle, \Delta} -_{\vee R}$$

For Bilateral LP, we add Bilateral Excluded Middle, and, for Bilateral K3, we add Bilateral Explosion. Adding both, we get Bilateral CL Equivalently, for LP, we can add Out, and, for K3, we can add In, and, once again, adding both, we get CL.[13] There are three crucial points about these system that deserve emphasis.

The first crucial point is that all of the rules in this system are *separable*. The inferential behavior of negation is given only by the negation rules, not by rules codifying its interaction with other connectives. Moreover, adding any set of rules to the fragment of the sequent calculus not containing those rules constitutes a conservative extension.[14] Separability is widely taken to be key formal constraint, on a par with harmony, in the context of logical inferentialism.[15] If rules are *not* separable—if, for instance, the rules for conjunction are not separable from the rules for negation—then, if we take seriously the that idea knowing the meaning of a connective is mastering the rules governing its use, it would seem that one could not know the meaning of negation without knowing the meaning of conjunction, nor could one know the meaning of conjunction without knowing the meaning of negation. As I explained above, the lack of separable rules for the negation in standard proof systems for FDE is, I think, the main reason that leads Beall to his conclusion that there is no logical negation. This system, in which there are separable rules that precisely characterize the inferential behavior of negation, completely undercuts that reason.

The second closely related crucial point is that the bilateral principles that, added to the sequent calculus for FDE, yield LP or K3 are *not* negation rules. They are, once

[13]It should be easy to see that these claims are true. However, the full proofs are provided in [reference omitted]. The admissibility of Cut, Weakening, and the eliminability of non-atomic instances of Reflexivity are also proven for these systems there

[14]This is notably not the case for the negation rules of standard unilateral natural deduction systems for classical logic, as evidenced by tautologies such as Peirce's law, which contains only the conditional yet is not provable in negation-free implicative fragment, given the usual conditional rules. In the sequent calculus setting, conservativity is proven by the proof of the admissibility of Cut.

[15]For a recent discussion of separability in the context of logical inferentialism, see [23].

again, *coordination principles*, bilateral structural rules that "coordinate" the relation between the speech acts of assertion and denial.[16] Beall's thought, transferred into this setting, is that, insofar as logic is maximally topic-neutral, and, in paradoxical contexts these coordination principles can be called into question, *logic itself* does not impose such coordination. At least for the sake of the present paper, I will grant this thought. But to say this is to say *nothing* about negation, since the crucial bilateralist thought is that coordination principles are *not* negation rules; they are distinctively bilateral *structural* rules. Explosion and Excluded Middle can, of course, be expressed with negation. For instance, we can express Explosion using negation as $+\langle A\rangle, +\langle\neg A\rangle \vdash +\langle B\rangle$. However, to think that, because of this it should be understood, fundamentally, as a principle about *negation* is a mistake. For instance, analogously, just because we can express Explosion as $+\langle A \wedge \neg A\rangle \vdash +\langle B\rangle$ does not mean that it's a principle about *conjunction*. Just as it's a mistake to talk about the distinction between "LP conjunction" and "Classical conjunction" on the basis that LP rejects this principle and Classical Logic accepts it, so too is it a mistake to talk about the distinction between "LP negation" and "Classical negation."

This brings us to the final crucial point, which is that, just as the conjunction rules are the same in each logic, so too, the negation rules are *exactly the same* whether one endorses FDE, LP, K3, or CL. Negation is a logical operator such that $\neg A$ is to be asserted just in case A is to be denied, and $\neg A$ is to be denied just in case A is to be asserted. That is what logical negation does; it toggles between a sentence's *truth*, its being correct to *assert*, and a sentence's *falsity*, its being correct to *deny*. I contend, then, that there is a logical negation, and it just is the logical operator that does just that.

5 Two Dimensional Bilateralism

I have laid out bilateral proof systems for the logics in the FDE family according to which the rules for negation are just those of bilateral proof systems for classical logic: negation is an operator such that, for any sentence A, asserting $\neg A$ is inferentially equivalent to denying A, and denying $\neg A$ is inferentially equivalent to asserting A. Thus, insofar as bilateral systems for classical logic can be understood as providing an account of logical negation in terms of its inferential role, these bilateral systems for the logics in the FDE can be understood as showing that the very same negation operator is operative in the context of these subclassical systems. All that differs between these systems is the principles "coordinating" the opposite acts of assertion and denial, establishing them as exhaustive, exclusive, or both.

This move to a bilateral setting should come as extremely natural to the proponent

[16] It is notable here that the coordination principles of the multiple conclusion systems can be completely restricted to atomics (see [omitted1]), and in the single conclusion system the oppositely signed formulas can be restricted to atomics (see the appendix, and [omitted2] for an argument that this is sufficient for "bilateral harmony" in natural deduction systems, which extends to this case).

of FDE. Still, the idea that one might reject the exclusivity of assertion and denial is likely to be met with resistance from bilateralists, who have typically assumed that, minimally, assertion and denial must be incompatible. This assumption can seem essential to the whole bilateralist idea of tying negation to denial, going back to Price [25]. Indeed, from a normative inferentialist perspective, it can be hard to see what denial *could even be*, if it is not incompatible with assertion. Of course, if one simply identifies the act of denial as the act of committing oneself to the falsity of a sentence (which is distinct from but inferentially equivalent to the act of committing oneself to the truth of its negation), then, if one is open to paraconsistency at all, it can be hard to see the cause for protest. But these characterizations of assertion and denial, which appeal to the alethic notions of truth and falsity, are not immediately available to the inferentialist.[17] An inferentialist, proposing a *use theory of meaning* can only appeal to aspects of the practice in which linguistic expressions are used. For instance, on the sort of inferentialist framework developed by Brandom ([10] [14]), assertion and denial can be understood as opposing "moves" in what Brandom [10] speaks of as "the game of giving and asking for reasons," and it can be hard to see what these opposing moves could be if they're not, minimally, incompatible such that performing one act precludes one from being entitled to perform the other. It thus worth developing this conception of assertion and denial, in the context of a normative inferentialist approach to content, in some detail.

I take as my starting point the version of the inferentialist framework recently put forward by Hlobil and Brandom [14]. In this framework, there are two fundamentally different sorts of reason relations: reasons *for* and reasons *against*. There are also two fundamentally different sorts of speech acts for which there may be reasons for or reasons against: *asserting* and *denying*. In the normal contexts that Hlobil and Brandom consider, reasons *for asserting* completely align with with reasons *against denying*, and reasons *for denying* completely align with reasons *against asserting*. However, Brandom himself at least acknowledges the possibility that, in paradoxical cases, these two dimensions might come apart. He says, for instance, "One might think that it is criterial of paradoxical sentences such as the Liar that subjects end up rationally committed *both* to accepting *and* to rejecting them, or that they are paradigms of sentences rational subjects should endeavor *neither* to accept *nor* reject," (54). So, one might have (all things considered) reasons *for asserting* but also have (all things considered) reasons *for denying*, or one might have (all things considered) reasons *against asserting* but also have (all things considered) reasons *against denying*. This is precisely the pair of possibilities, already implicit in Hlobil and Brandom's distinguishing these two dimensions, that this more flexible bilateral framework enables us to explore.

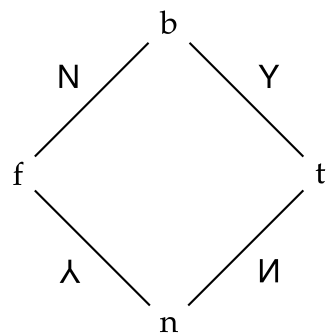
Though Brandom and Hlobil do not pursue this line of thought, the idea of these two dimensions (reasons for/reasons against, on the one hand, and assertion/denial, on the other) coming apart in this way has been explored in a technical semantic

[17] See especially Simonelli [41, 13-16] on this point.

context by Blasio, Marcos, and Wansing [9]. I will not go into the technical details of their approach, but, rather, focus on the conceptual underpinnings, which are shared by the proof-theoretic approach developed here.[18] Thinking of these two different dimensions of distinction, we arrive at four different types of reasons relations towards speech acts, which they label in the following way:

	Asserting	**Denying**
Reasons For	Y	N
Reasons Against	⅄	И

The truth-values *true*, *false*, *both*, and *neither*, can be understood as arising at the *intersections* of these four types of reason relations, as depicted in the following diagram:



To explicate this basic conception proof-theoretically, it will be illuminating to make this proof system *doubly bilateral*, such that, in addition to having signs for the acts of *asserting* and *denying*, we also signs for the acts of *making* a discursive move (an act underwritten by *reasons for*) and *challenging* a discursive move (an act underwritten by *reasons against*). Thus, we can distinguish the following four speech acts, each rationally underwritten by the corresponding reason relation stated above:

	the Assertion of A	**the Denial of A**
Making	$✓_+\langle A\rangle$	$✓_-\langle A\rangle$
Challenging	$✗_+\langle A\rangle$	$✗_-\langle A\rangle$

So, $✓_+\langle A\rangle$ and $✓_-\langle A\rangle$ express the speech acts of *asserting* A and *denying* A (the acts of *making* these opposite discursive moves), whereas $✗_+\langle A\rangle$ and $✗_-\langle A\rangle$ respectively express acts of *challenging* these opposite discursive moves.[19] One can think of the

[18] At the most basic level, they consider a semantics for "two-dimensional" consecutions of the form, $\frac{X_{1,1}}{X_{2,1}} \mid \frac{X_{1,2}}{X_{2,2}}$, which are valid, relative to a set of valuations V, just in case there's no $v \in V$ such that $0 \notin v(A)$ for all $A \in X_{1,1}$, $1 \notin v(B)$ for all $B \in X_{1,2}$, $1 \in v(C)$ for all $C \in X_{2,1}$, and $0 \in v(D)$ for all $D \in X_{2,2}$. It is easy to see that this corresponds to a bilateral sequent of the the form $+\langle X_{2,1}\rangle, -\langle X_{2,2}\rangle \vdash +\langle X_{1,2}\rangle, -\langle X_{1,1}\rangle$.

[19] Rather than double-signing formulas in this way, one can achieve the same effect by signing the turnstile and taking the left context to be a pair of sets of signed formulas, as Ayhan and Wansing.

act of challenging a move as an act in which one explicitly registers one's *opposition* to that move.[20]

Insofar as both gappy and glutty approaches to paradoxes such as the liar are under consideration, challenging the assertion of some sentence does not logically commit one to denying that sentence nor does denying some sentence logically commit one to challenging the assertion of that sentence. In this doubly bilateral framework, these thoughts can be made formally precise. Above I considered the coordination principles "In" and "Out," as principles relating assertions and denials. In, for instance, tells us that if one is committed to asserting A, then denying A is incoherent, whereas Out tells us that if it's incoherent to assert A, then one is committed to denying A. Once again, Bilateral LP rejects In but accepts Out, Bilateral K3 rejects Out but accepts In, and Bilateral FDE rejects both. *All* logics, however, can accept the following versions of In and Out relating the making and challenging of moves. For lack of a better term, I'll refer to such principles as *pragmatic* In and Out. Consider first pragmatic In (where φ is an assertion or denial):

$$\frac{\Gamma \vdash \checkmark\varphi}{\Gamma, \text{✗}\varphi \vdash}\ \text{p-In}_1 \qquad\qquad \frac{\Gamma \vdash \text{✗}\varphi}{\Gamma, \checkmark\varphi \vdash}\ \text{p-In}_2$$

These principles say that if Γ commits one to making some move φ (be it an assertion or denial), then Γ along with challenging φ is incoherent. Likewise, if Γ commits one to challenging φ, then Γ along with making the move φ is incoherent. Clearly, this seems right. Insofar as a challenge to a move functions to undermine the entitlement one has to that move, if one is committed to some move, then it is clearly incoherent to challenge that move, and if one is committed to challenging some move, then it is clearly incoherent to make a move. Now consider the pragmatic Out rules:

[20]The acts of challenging an assertion and challenging the denial, expressed here with $\text{✗}_+\langle A\rangle$ and $\text{✗}_-\langle A\rangle$, correspond closely to the acts Incurvati and Schlöder [17] [15] [16] term "weak rejection" and "weak assertion," expressed in their multilateral system with $\ominus\langle A\rangle$ and $\oplus\langle A\rangle$ (though the logic is not quite the same, as theirs has a distinctively epistemic flavor). Putting a bilateralist twist on Stalnaker [43], Incurvati and Schlöder understand these acts in terms of their potential to change the common ground, understood as partitioned into the *positive* common ground and the *negative* common ground. The aim of asserting a sentence is to include it in the positive common ground, and the aim of denying a sentence is to include it in the negative common ground. The aim of challenging the assertion of a sentence ("weakly denying" it) is to exclude it from the positive common ground, and the aim of challenging the denial of a sentence ("weakly asserting" it) is to exclude it from the negative common ground. The key difference is that Incurvati and Schlöder suppose that there can be no overlap in sentences included in the positive and negative common ground [16, 60-70], whereas the framework here permits such overlaps. As a result, the use of their terminology would be misleading in this context, since, as we'll see shortly, in Bilateral LP, "weak denial" is in fact strictly stronger than "strong denial." I thus opt to speak simply in terms of making and challenging assertions and denials, finding it more perspicuous to deploy a notation that reflects this terminological choice.

$$\frac{\Gamma, \checkmark\varphi \vdash}{\Gamma \vdash \text{✗}\varphi}\ \text{p-Out}_1 \qquad\qquad \frac{\Gamma, \text{✗}\varphi \vdash}{\Gamma \vdash \checkmark\varphi}\ \text{p-Out}_2$$

These principles say that if Γ along with making the move φ is incoherent, then Γ commits one to challenging φ. Likewise, if Γ along with challenging φ is incoherent, then Γ commits one to making the move φ.

It is indeed trivial to expand the notion of correctness to apply to formulas of the form ✗φ in such a way that validates p-In and p-Out: ✗φ is correct, relative to a valuation v, just in case φ is incorrect, relative to v.[21] We can show, now that, in Bilateral K3 denying some sentence commits one to challenging the assertion of that sentence, but not vice versa, whereas, in Bilateral LP, challenging the assertion of some sentence commits one to denying that sentence, but not vice versa. For the positive part of this claim, just consider the following two proofs:

$$\dfrac{\dfrac{\dfrac{}{\checkmark\text{-}A \vdash \checkmark\text{-}A}\ \text{Reflex.}}{\checkmark\text{-}A, \checkmark\text{+}A \vdash}\ \text{In}}{\checkmark\text{-}A \vdash \text{✗+}A}\ \text{p-Out} \qquad\qquad \dfrac{\dfrac{\dfrac{}{\checkmark\text{+}A \vdash \checkmark\text{+}A}\ \text{Reflex.}}{\checkmark\text{+}A, \text{✗+}A \vdash}\ \text{p-In}}{\text{✗+}A \vdash \checkmark\text{-}A}\ \text{Out}$$

To see that, in each case, the converse doesn't hold, just note that, if it did, we could derive Out in BK3 and In in BLP.

Finally, we can now note explicit that the principles I've called "In" and "Out" are principles relating the *making* of assertions and denials On the flip side, we can now consider "In" and "Out" relating the *challenging* of assertions and denials:

$$\frac{\Gamma \vdash \text{✗}\varphi}{\Gamma, \text{✗}^{*}\varphi \vdash}\ \text{c-In} \qquad\qquad \frac{\Gamma, \text{✗}\varphi \vdash}{\Gamma \vdash \text{✗}\varphi^{*}}\ \text{c-Out}$$

Whereas, pertaining to the making of moves, Bilateral LP accepts Out but rejects In and Bilateral K3 accepts In but rejects Out, pertaining to the challenging of moves, the exact opposite holds. For instance, the proponent of Bilateral K3 that takes a gappy sentence takes it that one is committed to challenging the assertion of the liar, but that doesn't mean that they take it to be incoherent to challenge it's denial. On the contrary, such a logician takes it that one is committed to challenging its denial too. Dually, the proponent of Bilateral LP takes it to be incoherent to challenge the assertion liar sentence (as they take one to be committed to making this assertion), but that doesn't mean that they take one to be committed to challenging the denial of the liar. On the contrary, such a logician takes it that one is committed to asserting its denial too.

It seems clear that considering these dual gappy and glutty possibilities, in which assertion and denial are not assumed to be exhaustive or exclusive, makes good

[21]Given this way of defining correctness for doubly bilateral formulas, it is equally trivial that to show that the expanded proof system, which adds p-In and p-Out, is sound and complete relative to doubly bilateral consequence, and that the above points about the separability of the negation rules hold for this expanded system.

sense, and this is only possible insofar as logic itself imposes neither the exhaustivity nor the exclusivity of assertion and denial. Beyond further clarifying these possibilities, making this two-dimensional bilateral approach explicit in the notation enables us to respond to a number of other concerns that one might have about this approach. I'll focus on two additional concerns that might seem most pressing.

The first additional concern has to do with the fact that bilateralist conception of assertion and denial I've appealed to here in putting forth these paraconsistent systems, according to which denying a sentence is inferentially equivalent to asserting its negation (and thus, a dialetheist ought to both assert and deny a sentence they take to be true and false), is out of line with the use of "assertion" and "denial" by prominent dialetheists such as Priest [28] [29]. Consider, for instance, what Priest [28] says about the assertions and denials made by gap-theorists and glut-theorists:

> Consider someone who supposes that some sentences are neither true nor false. Let A be a sentence that they take to be of this kind. They will then deny A; but their denial is certainly not to be taken as an assertion of $\neg A$. [. . .] Conversely, a dialetheist who has ground for believing that A and $\neg A$ are both true may assert $\neg A$ without thereby denying A, (104-105).

This is out of line with the approach I've laid out here. On my approach, the gap theorist to who takes A to be neither true nor false should *not* deny A (nor should they assert it), and the glut theorist who takes A to be both true and false *should* deny A (as well as asserting it). One might wonder, then, if the notions of "assertion" and "denial" at use in the paraconsistent systems put forward here are not tracking the use of those notions by actual paraconsistent logicians, what reason do we have to use such systems to develop paraconsistent theories?

In fact, however, we can now show that this framework enables us to define a notion of denial that precisely tracks the way in which Priest uses the notion. In particular, this framework enables us to distinguish between two senses of the "denial" of A. The first sense of "denial," which we may denote *denial*$_1$, is the sense expressed here by ✓-A, which is inferentially equivalent to (though not identical to) the assertion of $\neg A$. The second sense of "denial," which we may denote *denial*$_2$, is the sense expressed here by ✗+A, whose performance amounts to challenging the assertion of A.[22] We showed above that, in Bilateral K3, denying$_1$ A is strictly stronger than denying$_2$ A in that performing the former act commits one to performing the latter but not vise versa, whereas, in Bilateral LP, the exact converse holds. Given our distinction between the two senses of "denial," and our formalization of the inferential norms governing "denial" in these respective senses, it is clear that, by "denial," Priest means denial$_2$. That is, he means the act of *opposing* an assertion. This disambiguation between senses of "denial" shows that any disagreement about the nature of assertion and denial between Priest and a proponent of the bilateral systems laid out here is merely verbal. There is *a* use of "denial" such that Priest's

[22] This, once again, is a version of what Incurvati and Schloder call "weak denial."

statement quoted above is perfectly correct: the one expressed here by $\text{✗}_{+}A$. It's just that the term "denying A" is principally used here to express the act of taking A to be false, expressed here with $\text{✓}_{-}A$.[23]

A further as of yet unaddressed issue has to do with the fact that among the bilateral systems put forward here are *multiple conclusion* sequent calculi. Above, when explicating the significance of the negation rules of the familiar *uni*lateral multiple conclusion sequent calculus, I appealed to Restall's bilateral interpretation of multiple conclusion sequents, according to which $X \vdash Y$ is read as saying that asserting everything in X and denying everything in Y is incoherent or "out of bounds." This interpretation is widely appealed to by proponents of inferentialism using multiple conclusion sequents in formally developing their inferentialist theories.[24] However, in the explicitly bilateral framework put forward here, we have *bilateral* multiple conclusion sequents of the form $\Gamma \vdash \Delta$ where the formulas in Γ and Δ are *themselves* assertions and denials. Since assertions and denials cannot themselves be asserted or denied, Restall's bilateral reading of these multiple conclusion sequents is unavailable. Accordingly, it's still not clear that one can appeal to these multiple conclusion systems in the context of an inferentialist theory.

However, though the specific bilateral reading of multiple conclusion sequents proposed by Restall is unavailable in this explicitly bilateral context, the same general sort of "bounds consequence" (Ferguson [12]) conception is nevertheless straightforward available. We can read $\Gamma \vdash \Delta$ as saying that *making* all of the moves in Γ and *challenging* all of the moves in Δ is "out of bounds." Thus, the bilateral system can be interpreted essentially in Restall-style bilateralist fashion. The axiom of Reflexivity, for instance, amounts to the thought that making some move (be it an assertion or denial) and challenging that very move is always incoherent. Though the proponent of Bilateral LP does not think that asserting and denying the same sentence must be incoherent, they surely do think that asserting some sentence and challenging that very assertion is (indeed, that's just Priest's point, stated above). Indeed, beyond just interpreting this sequent calculus bilaterally in this way, we can use the doubly bilateral system just above to make this interpretation explicit in the notation itself, just as we made the Restall's bilateral interpretation of unsigned sequents explicit in the notation. Thus, we have the following translation schema:

> **Translation Schema (Round 2):** To translate an bilateral multiple conclusion sequent of the form $\Gamma \vdash \Delta$ to doubly bilateral sequent of the form

[23]It does not seem particularly fruitful to ask which of these two senses of "denial" more closely corresponds to the standard use of the term in ordinary English. Clearly, in ordinary contexts, these two notions are blurred together, as, in all normal circumstances, explicitly taking some sentence to be false is tantamount to disagreeing with anyone who takes that sentence to be true. However, in various odd contexts that are of interest to philosophers and logicians, these notions can come apart, and we can systematically investigate the ways in which they can come apart and are related in this framework.

[24]See, for instance, [34], [35], [46], and Brandom and Hlobil.

$\Theta \vdash$, let $\Theta = \{✓\varphi \mid \varphi \in \Gamma\} \cup \{✗\psi \mid \psi \in \Delta\}$.

Applying this translation schema, one can rewrite the above provided sequent calculus such that it features only solely left-sided sequents, encoding incoherence. Eech such sequent corresponds to an equivalence class of single conclusion sequents. Letting $\Phi^\star$ denote the *pragmatic opposite* of Φ ($✗\varphi$ if Φ is of the form $✓\varphi$ and $✓\varphi$ if Φ is of the form $✗\varphi$), given Pragmatic In and Out, such doubly bilateral solely-left-sided sequents of the form $\Theta \vdash$ correspond to an equivalence class of sequents of the form $\{\Theta/\Phi\} \vdash \Phi^\star$ for all $\Phi \in \Theta$. Given these equivalences, we can understand, for instance, the positive disjunction right rule:

$$\frac{\Gamma \vdash +\langle A\rangle, +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \vee B\rangle, \Delta}\ +_{\vee_R}$$

As equivalent to the following rules, which are all equivalent to one another:[25]

$$\frac{\Theta, ✗_+\langle A\rangle, ✗_+\langle B\rangle \vdash}{\Theta \vdash ✓_+\langle A \vee B\rangle}\ +_{\vee_R} \qquad \frac{\Theta, ✗_+\langle A\rangle \vdash ✓_+\langle B\rangle}{\Theta \vdash ✓_+\langle A \vee B\rangle}\ +_{\vee_R} \qquad \frac{\Theta, ✗_+\langle B\rangle \vdash ✓_+\langle A\rangle}{\Theta \vdash ✓_+\langle A \vee B\rangle}\ +_{\vee_R}$$

That is, it tells us that one is committed to asserting $A \vee B$, given one's set of moves made and moves challenged, just in case challenging the assertion of A along with challenging the assertion of B is incoherent, or equivalently, just in case challenging the assertion of one of the disjuncts commits one to making the assertion of the other. In this way, even the multiple conclusion sequent calculus can be understood as providing inferential semantic clauses for the connectives in terms of making and challenging assertions and denials.

6 Another Negation?

I take myself to have shown that there is a logical negation, even on the FDE picture of logic. But showing that there is *a* logical negation does not establish that there is *only one* logical negation. Logical negation, in the sense I have codified, expresses denial: asserting $\neg A$ is *distinct from* but *inferentially equivalent to* denying A. However, as I've argued above, when some authors, such as Priest, speak of "denial," they mean this not in the sense of *committing oneself to the falsity* of a sentence, but, rather, in the sense of *opposing the commitment to the truth* of a sentence. Here, again, is Priest [29]on his notion of denial:

> Suppose that you assert A. There is nothing I can *assert* that entails disagreement (as opposed to conversationally implicating it). But I can *deny* A, which will do the trick. Denial is a speech act distinct from asserting (like commanding or questioning); and, post-Fregean wisdom to the contrary, it is sui generis, not to be reduced to asserting the negation of A, (291-292).

[25] See [40] for a defense of rules of this form in a singly bilateral context.

Priest says "there is nothing I can assert that entails disagreement." But why not? Insofar as we can formally codify the inferential norms governing disagreement, why can we not introduce a special negation operator $\sim$, such that asserting $\sim A$ is distinct from but inferentially equivalent to disagreeing with (i.e. challenging) the assertion of A? Well, let's see.

It's clear what the rules for asserting a sentence containing such an operator should be, as well as the rules for challenging such an assertion:

$$\frac{\Theta \vdash \times_{+} A}{\Theta \vdash \checkmark_{+} \sim A} \qquad\qquad \frac{\Theta \vdash \checkmark_{+} A}{\Theta \vdash \times_{+} \sim A}$$

There are a few different ways to give rules for denial; however, the most proof-theoretically natural way, in the doubly bilateral setting I've introduced, is to take $\sim$ to be another "flip-flip" operator. That is, whereas $\neg$ flip-flops between *assertion* and *denial*, $\sim$ functions to flip-flop between *making* and *challenging*, such that making an assertion (or denial) of $\sim A$ is correct just in case challenging an assertion (or denial) of A is correct, and challenging an assertion or denial of $\sim A$ is correct just in case making an assertion (or denial) of A is correct. So, we can consider an operator with the above rules for asserting and the following rules for denial:

$$\frac{\Theta \vdash \times_{-} A}{\Theta \vdash \checkmark_{-} \sim A} \qquad\qquad \frac{\Theta \vdash \checkmark_{-} A}{\Theta \vdash \times_{-} \sim A}$$

Whether or not there is in fact an English expression that plays this inferential role, it seems completely reasonable to introduce into our formal language a negation operator that does. However, if our language contains a transparent truth-predicate enables the construction of liar-sentences, we cannot introduce such a negation into it without triviality. Just consider the sentence ℓ, defined as $\sim T\langle \ell \rangle$. Such a sentence trivializes the consequence relation as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\checkmark_{+}\ell \vdash \checkmark_{+}\ell}\,\text{Reflexivity}}{\checkmark_{+}\ell \vdash \checkmark_{+}T\langle\ell\rangle}\,\text{tTruth}}{\checkmark_{+}\ell \vdash \times_{+} \sim T\langle\ell\rangle}\,\times_{+\sim}}{\checkmark_{+}\ell \vdash \times_{+}\ell}\,\text{meaning of } \ell}{\checkmark_{+}\ell, \checkmark_{+}\ell \vdash}\,\text{p-In}}{\checkmark_{+}\ell \vdash}\,\text{Contraction}}{\vdash \times_{+}\ell}\,\text{p-Out} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\times_{+}\ell \vdash \times_{+}\ell}\,\text{Reflexivity}}{\times_{+}\ell \vdash \times_{+}T\langle\ell\rangle}\,\text{tTruth}}{\times_{+}\ell \vdash \checkmark_{+} \sim T\langle\ell\rangle}\,\checkmark_{+\sim}}{\times_{+}\ell \vdash \checkmark_{+}\ell}\,\text{meaning of } \ell}{\times_{+}\ell, \times_{+}\ell \vdash}\,\text{p-In}}{\times_{+}\ell \vdash}\,\text{Contraction}}{\vdash}\,\text{Cut}$$

Using just these principles, we conclude that, even without making or challenging any assertions or denials, one is *committed* to challenging the assertion of ℓ, and yet, challenging the assertion of ℓ is *incoherent*. So, the null position, without making or challenging any assertions or denials, is incoherent. And, of course, given Weakening, everything is incoherent and everything implies everything else.

It may come as no surprise that these are just the rules, in this doubly bilateral framework, for *Boolean Negation*, the semantic clause of which is as follows:[26]

$$v(\sim A) \ni \begin{cases} 1, & \text{if } v(A) \not\ni 1 \\ 0, & \text{if } v(A) \not\ni 0 \end{cases}$$

The fact that the dialethic approaches to the liar cannot add Boolean Negation into their theory is well-known and well-discussed.[27] However, this new proof-theoretic context provides a new spin on this old problem. According to Priest [27]:

> If Boolean negation is characterised proof-theoretically, it is certainly inexpressible (on pain of triviality). However, in this case it cannot be shown to have determinate sense, (290).

It seems hard to maintain, in this context, that $\sim$ really *does* lack a determinate sense. First, though Priest notes the familiar point from Prior's [32] *tonk* (and Belnap's [7] *plonk*) that it isn't the case that merely any set of rules counts as defining a legitimate connective, the rules for $\sim$ meet all of the proof-theoretic constraints that have been proposed to rule out such connectives. Indeed, from the perspective of proof-theoretic semantics, the rules for $\sim$ are *perfect*: harmonious (unilaterally and bilaterally), separable, invertible—*you name it!*[28] Secondly, and more importantly, these rules are not mere formal stipulation, but, rather, seem to fall out naturally from the attempt to express the concepts that Priest himself seems to use in articulating the dialethic position, according to which one might coherently assert both some sentence and its negation, opposing neither such assertion.

Thus, though I've articulated this bilateral approach to paraconsistent logics on grounds quite sympathetic to the dialethic program, this framework nevertheless crystallizes a longstanding objection to the dialethic response to paradox. I am not sure how the dialetheist ought to respond. Whatever they say, however, it seems clear that they're committed to saying that $\neg$, an account of which we've given here, is logical negation: the one and only.

[26]In the multiple conclusion bilateral framework, the rules are the following:

$$\frac{\Gamma, +\langle A\rangle \vdash \Delta}{\Gamma \vdash +\langle \sim A\rangle, \Delta}\ +_{\sim_R} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta}{\Gamma, +\langle \sim A\rangle \vdash \Delta}\ +_{\sim_L} \qquad \frac{\Gamma - \langle A\rangle \vdash \Delta}{\Gamma \vdash -\langle \sim A\rangle, \Delta}\ -_{\sim_R} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta}{\Gamma, -\langle \sim A\rangle \vdash \Delta}\ -_{\sim_L}$$

[27]See, for instance [8], [27], [38].

[28]Of course, as Priest notes, they are not conservative over a language that contains a transparent truth-predicate and self-reference, but that was never the sort of language in which conservativity was introduced as a harmony constraint. Conservativity, relative to a *logical* language, is best understood as a *consequence* of harmony, not a *criterion* of it.

7 Technical Appendix

7.1 On Single Conclusion Bilateral Systems

I just treat here the key properties of the Bilateral Single-Conclusion Sequent (BSS) systems, which are novel to this paper. My treatment of the multiple conclusion sequent systems can be found in [reference omitted].

Soundness of $\mathbf{BSS_{FDE}}$: If $\mathrm{BSS_{FDE}}$ proves $\Gamma \vdash \varphi$, then $\Gamma \vDash_{B_{\mathrm{FDE}}} \varphi$

Proof: Straightforward by induction on proof height. Clearly, any instance of Reflexivity is valid, and it's simple to show that the rules preserve validity. □

Completeness of $\mathbf{BSS_{FDE}}$: If $\Gamma \vDash_{B_{\mathrm{FDE}}} \varphi$, then $\mathrm{BSS_{FDE}}$ proves $\Gamma \vdash \varphi$.

Proof: I'll just sketch the standard Henkin-style proof, which proceeds as usual in this bilateral setting.[29] We prove the contrapositive, supposing $\Gamma \nvdash \varphi$ and constructing an FDE counterexample to show that $\Gamma \nvDash_{B_{\mathrm{FDE}}} \varphi$.

We first construct a saturated set $\Delta \supseteq \Gamma$ such that Δ is deductively closed (in that if $\Delta \vdash \psi$, given the rules of the calculus, then $\psi \in \Delta$) and $\Delta \nvdash \varphi$. To do this, we enumerate the formulas of the language ψ_i for $i \in \omega$. Now we define Δ as follows:

1. $\Delta_0 = \Gamma$
2. $\Delta_{n+1} = \Delta_n \cup \{\psi_n\}$ if $\Delta_n \cup \{\psi_n\} \nvdash \varphi$; $\Delta_{n+1} = \Delta_n$ otherwise.
3. $\Delta = \bigcup_{n \in \omega} \Delta_n$

It is easy to see Δ meets the above stated condition. Clearly, $\Delta \supseteq \Gamma$. By induction on n, each n is such that $\Delta_n \nvdash \varphi$, and so, by compactness, $\Delta \nvdash \varphi$. To see that Δ is deductively closed, suppose for reductio that $\Delta \vdash \psi_n$ and $\psi_n \notin \Delta$. Then $\Delta_n \cup \{\psi_n\} \vdash \varphi$. Contradiction, so Δ is deductively closed.

We now define a valuation v such that, for all sentences A, $1 \in v(A)$ just in case $+\langle A \rangle \in \Delta$ and $0 \in v(A)$ just in case $-\langle A \rangle \in \Delta$, and we show by induction on the complexity of the formulas in Δ that this is an FDE valuation. The base case is immediate, since FDE permits any assignment of values to atomics. For the inductive step, we show how the deductive closure of Δ ensures that v conforms the the semantic clauses. Consider the case of negation. We must show that $1 \in v(\neg A)$ just in case $0 \in v(A)$ and $0 \in v(\neg A)$ just in case $1 \in v(A)$. If $0 \in v(A)$, then, by the definition of v, $-\langle A \rangle \in \Delta$. By the Reflexivity and the positive negation right rule, $\Delta, -\langle A \rangle \vdash +\langle \neg A \rangle$, and so, given that Δ is deductively closed, $+\langle \neg A \rangle \in \Delta$, and thus $1 \in v(\neg A)$. If $0 \notin v(A)$, then $-\langle A \rangle \notin \Delta$. By Reflexivity and the positive negation left rule, $\Delta, +\langle \neg A \rangle \vdash -\langle A \rangle$, and so, given deductive closure, $+\langle \neg A \rangle \notin \Delta$, and thus $1 \notin v(\neg A)$. Consideration of the $-_\neg$ rules establish that $0 \in v(\neg A)$ just in case $1 \in v(A)$, and similar considerations regarding the conjunction and disjunction rules establish v conforms to the semantic clauses for conjunction and disjunction. So, we have an

[29] See [31] for a similar proof in the context of unilateral natural deduction for FDE.

FDE valuation v such that all the formulas in Δ are correct. But since $\varphi \notin \Delta$ (since, if it was, we'd have $\Delta \vdash \varphi$), φ is incorrect. So, $\Delta \nvDash_{B_{\text{FDE}}} \varphi$, and since $\Delta \supseteq \Gamma$, $\Gamma \nvDash_{B_{\text{FDE}}} \varphi$. □.

Soundness of $\mathbf{BSS_{LP}}$: If BSS_{LP} proves $\Gamma \vdash \varphi$, then $\Gamma \vDash_{B_{\text{LP}}} \varphi$

Proof: Simple to show that Bilateral Excluded Middle preserves validity over LP valuations. □

Completeness of $\mathbf{BSS_{LP}}$: If $\Gamma \vDash_{B_{\text{LP}}} \varphi$, then BSS_{LP} proves $\Gamma \vdash \varphi$.

Proof: Same as before, but we construct Δ in such a way that it is closed under the rules of BSS_{LP}, which additionally include Bilateral Excluded Middle. Here, note that, for every sentence A, Δ must contain either $+\langle A\rangle$ or $-\langle A\rangle$. Suppose Δ contained neither. This could only be because $\Delta, +\langle A\rangle \vdash \varphi$ and $\Delta, -\langle A\rangle \vdash \varphi$. But then, given that Δ is deductively closed, by BExM, $\Delta \vdash \varphi$. Contradiction. Given that Δ must contain either $+\langle A\rangle$ or $-\langle A\rangle$, for any sentence A, either $1 \in v(A)$ or $0 \in v(A)$, and thus the valuation v, defined as above, is an LP valuation. □.

Soundness of $\mathbf{BSS_{K3}}$: If BSS_{K3} proves $\Gamma \vdash \varphi$, then $\Gamma \vDash_{B_{\text{K3}}} \varphi$

Proof: Simple to show that Bilateral Explosion preserves validity over K3 valuations. □

Completeness of $\mathbf{BSS_{K3}}$: If $\Gamma \vDash_{B_{\text{K3}}} \varphi$, then BSS_{K3} proves $\Gamma \vdash \varphi$.

Proof: Same as before, but we construct Δ in such a way that it is closed under the rules of BSS_{K3}, which additionally include Bilateral Explosion. Here, note that, for every sentence A, Δ cannot contain both $+\langle A\rangle$ and $-\langle A\rangle$. Suppose Δ contained both. Then, by reflexivity, $\Delta \vdash +\langle A\rangle$ and $\Delta \vdash -\langle A\rangle$. But then, given that Δ is deductively closed, by BExplo, $\Delta \vdash \varphi$. Contradiction. Given that Δ cannot contain both $+\langle A\rangle$ and $-\langle A\rangle$, for any sentence A, either $1 \notin v(A)$ or $0 \notin v(A)$, and thus the valuation v, defined as above, is an K3 valuation. □.

Admissibility of Cut and Weakening: Direct proofs can be provided, but it is sufficient to note that they are validity-preserving and the completeness proofs do not appeal to them. □

Restriction of Oppositely Signed formulas in BExplo and BExMid to atomics: In all logics containing them, the coordination principles of B. Explo and B. Ex. Mid. can be replaced with the following principles, where p is an atom:

$$\frac{\Gamma \vdash +\langle p\rangle \quad \Gamma \vdash -\langle p\rangle}{\Gamma \vdash \psi}\ \text{(m) B. Explo.} \qquad \frac{\Gamma, +\langle p\rangle \vdash \psi \quad \Gamma, -\langle p\rangle \vdash \psi}{\Gamma \vdash \psi}\ \text{(m) B. Ex. Mid.}$$

where $\{\psi\}$ can be null

Proof: I'll restrict my attention to conjunction here, as the case for negation is trivial and the case for disjunction is dual. First we need to establish the following:

Invertibility Lemma:

- If $\Gamma, +\langle A \wedge B\rangle \vdash \psi$ is derivable $\Gamma, +\langle A\rangle, +\langle B\rangle \vdash \psi$ is derivable.
- If $\Gamma, -\langle A \wedge B\rangle \vdash \psi$ is derivable, $\Gamma, -\langle A\rangle \vdash \psi$ is derivable and $\Gamma, -\langle B\rangle \vdash \psi$

- If $\Gamma \vdash +\langle A \wedge B \rangle$ is derivable, $\Gamma \vdash +\langle A \rangle$ is derivable and $\Gamma \vdash +\langle B \rangle$
- If $\Gamma \vdash -\langle A \wedge B \rangle$ is derivable, either $\Gamma \vdash -\langle A \rangle$ is derivable or $\Gamma \vdash -\langle B \rangle$

Proof: We induct on the height of a Cut-free derivation, supposing that the lemma holds at height n and showing that it holds at height $n+1$. If the conclusion of $\pm\langle A \wedge B \rangle$ is not the last step in the derivation, then all four clauses are immediate from the inductive hypothesis. If the conclusion of $\pm\langle A \wedge B \rangle$ is the last step in the derivation, then it is easy to see that all four clauses hold by inspection of the rules. □

We now show that, in any case that BExplo or BExMid is applied where the opposite formulas are complex, BExplo or BExMid can be applied on the simpler formulas in the premise sequents needed to derive those formulas to yield the same sequent. For B. Ex. Mid., we have the following reduction:

$$\dfrac{\Gamma, +\langle A \wedge B \rangle \vdash \psi \quad \Gamma, -\langle A \wedge B \rangle \vdash \psi}{\Gamma \vdash \psi}\ \text{B. Ex. Mid.}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, +\langle A \rangle, +\langle B \rangle \vdash \psi \quad \Gamma, -\langle A \rangle \vdash \psi}{\Gamma + \langle B \rangle \vdash \psi}\ \text{B. Ex. Mid.} \qquad \Gamma, -\langle B \rangle \vdash \psi}{\Gamma \vdash \psi}\ \text{B. Ex. Mid.}$$

For B. Explo, we have one of the following two reductions:

$$\dfrac{\Gamma \vdash +\langle A \wedge B \rangle \quad \Gamma \vdash -\langle A \wedge B \rangle}{\Gamma \vdash \psi}\ \text{B. Explo.}$$

$$\rightsquigarrow$$

$$\dfrac{\Gamma \vdash +\langle A \rangle \quad \Gamma \vdash -\langle A \rangle}{\Gamma \vdash \psi}\ \text{B. Explo.} \quad \textit{or} \quad \dfrac{\Gamma \vdash +\langle B \rangle \quad \Gamma \vdash -\langle B \rangle}{\Gamma \vdash \psi}\ \text{B. Explo.}$$

Since every such reduction reduces the complexity of the oppositely signed formulas, the process must terminate after finitely many steps, yielding a derivation that uses the coordination principles only on oppositely signed atomics. □

7.2 On Doubly-Signed Systems

I introduce doubly-signed formulas in Section 5 mainly for elucidatatory purposes, but I'll state some basic facts about these systems here. I will consider here just single conclusion consequence relations for the doubly-signed formulas introduced in Section 5. Officially, we can straightforwardly extend the notion of correctness to apply to doubly-signed formula as follows:

Correctness: Relative to some valuation v

1. $✓_+\langle A\rangle$ is correct just in case $1 \in v(A)$
2. $✓_-\langle A\rangle$ is correct just in case $0 \in v(A)$
3. $✗_+\langle A\rangle$ is correct just in case $1 \notin v(A)$
4. $✗_-\langle A\rangle$ is correct just in case $0 \notin v(A)$

We retain the same notion of validity, such that $\Theta \vdash \Phi$ is valid (where Θ is a set of doubly signed formulas and Φ is a doubly signed formula) just in case, there's no valuation v such that all of the stances in Θ are correct and Φ is incorrect. The following result is more or less immediate:

Completeness of Translated Multiple Conclusion Systems with Respect to Doubly Bilateral Validity: Consider the translation of the multiple conclusion bilateral system into a calculus relating solely left-sided doubly signed sequents, given the translation schema in Section 5, with the the coordination principles of p-In and p-Out.[30] This calculus is complete with respect to validity for doubly signed formulas.

Proof: It follows directly from these definitions that $\Gamma \vDash \Delta$ just in case $✓(\Gamma) \cup ✗(\Delta) \vDash$. Thus, given the completenes of the multiple conclusion bilateral system and the translation schema, the solely left-sided system is complete with respect to solely left-sided validities. Moreover, $\Theta \vDash \Phi$ just in case $\Theta, \Phi^\star \vDash$. Thus, for any valid sequent of the form $\Theta \vDash \Phi$, the solely left-sided system derives $\Theta, \Phi^\star \vdash$, and thus, via Out, $\Theta \vdash \Phi$. □

In addition to this solely-left-sided translation of the multiple conclusion system is straightforward to minimally modify the single conclusion system so that it extends to doubly signed formulas. Just take the contexts on the left to be sets of doubly signed formulas, the signed formula in the left rules and ψ in BExplo and BExmid to be an arbitrary doubly-signed formula, sign everything else with ✓, and add the rules p-In and p-Out. We may call such systems dBSS systems.

Completeness of dBSS Systems with Respect to Doubly Bilateral Validity: All dBSS systems are complete with respect to doubly-bilateral validity.

Proof: Consider first the system for FDE. Note first that, given right rules for making moves and p-In and p-Out, we can derive empty right-hand-sided left rules for challenges in the following way:

$$\dfrac{\dfrac{\dfrac{\Theta, ✗_+\langle A\rangle \vdash}{\Theta \vdash ✓_+\langle A\rangle}\,\text{p-Out} \qquad \dfrac{\Theta, ✗_+\langle B\rangle \vdash}{\Gamma \vdash ✓_+\langle B\rangle}\,\text{p-Out}}{\Theta \vdash ✓_+\langle A \wedge B\rangle}\,+_{\wedge_R}}{\Theta, ✗_+\langle A \wedge B\rangle \vdash}\,\text{p-In} \qquad \dfrac{\dfrac{\dfrac{\Theta, ✗_-\langle A\rangle \vdash}{\Theta \vdash ✓_-\langle A\rangle}\,\text{p-Out}}{\Theta \vdash ✓_-\langle A \wedge B\rangle}\,-_{\wedge_R}}{\Theta, ✗_-\langle A \wedge B\rangle \vdash}\,\text{p-In} \qquad \dfrac{\dfrac{\dfrac{\Theta, ✗_-\langle B\rangle \vdash}{\Theta \vdash ✓_-\langle B\rangle}\,\text{p-Out}}{\Theta \vdash ✓_-\langle A \wedge B\rangle}\,-_{\wedge_R}}{\Theta, ✗_-\langle A \wedge B\rangle \vdash}\,\text{p-In}$$

[30] Following the approach of [40], one could also consider an equivalent version of this system featuring only right rules of the sort.

Considering the left rules for making moves (with null right-hand sides) and derived left-rules for challenging moves of the above sort, we can not that the only rules of the translated multiple conclusion system that are not derivable in this system are the rules corresponding to the negative right conjunction and positive disjunction right rules. But, since Weakening is admissible, the rules belonging to the translated multiple conclusion systems are clearly admissible, and, so, given the completeness of that system for FDE, this system is complete.

For LP and K3, we can just note that the translation of the axioms of Bilateral Excluded Middle and Explosions can be derived as follows:

$$\dfrac{\dfrac{\dfrac{}{\Theta, \checkmark\varphi \vdash \checkmark\varphi}\text{Reflex.}}{\Theta, \text{✗}\varphi, \checkmark\varphi \vdash}\text{p-In} \qquad \dfrac{\dfrac{}{\Theta, \checkmark\varphi^* \vdash \checkmark\varphi^*}\text{Reflex.}}{\Theta, \text{✗}\varphi^*, \checkmark\varphi^* \vdash}\text{p-In}}{\Theta, \text{✗}\varphi, \text{✗}\varphi^* \vdash}\text{BExMid} \qquad \dfrac{\dfrac{}{\Theta, \checkmark\varphi \vdash \checkmark\varphi}\text{Reflex.} \qquad \dfrac{}{\Theta, \checkmark\varphi^* \vdash \checkmark\varphi^*}\text{Reflex.}}{\Theta, \checkmark\varphi, \checkmark\varphi^* \vdash}\text{BExplo}$$

□

Separability of the Negation Rules in Doubly-Singed Systems: As with the standard bilateral systems, all operational rules in these doubly-signed systems are *pure*, containing only a single connective, and adding any connective to a fragment of the system not containing that connective constitutes a *conservative extension* of the consequence relation generated by those rules, as established by Cut Elimination for the bilateral systems. □

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