“Yes,” “No,” Both, and Neither:
Bilateral Systems for the FDE Family

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Bilateralism

- **Bilateral Proof Systems (Smiley 1996, Rumfitt 2000):** Provide rules for both *affirming* and *denying* sentences, prefixing each sentence with a positive or negative force-marker to indicate its affirmation or denial.
  - $+\langle\varphi\rangle$ indicates the affirmation of φ
  - $-\langle\varphi\rangle$ indicates the denial of φ
  - **Note:** Signs are neither embedable nor iterable.

Well-Formed:	**Not Well-Formed**
$+\langle p \wedge \neg\langle q\rangle\rangle$	$+\langle p \wedge -\langle q\rangle\rangle$
$-\langle\neg p\rangle$	$-\langle -\langle p\rangle\rangle$

  - **Rumfitt's (2000) rules for negation:**

$$\frac{-\langle\varphi\rangle}{+\langle\neg\varphi\rangle}\ +_{\neg_I} \qquad \frac{+\langle\neg\varphi\rangle}{-\langle\varphi\rangle}\ +_{\neg_E}$$

$$\frac{+\langle\varphi\rangle}{-\langle\neg\varphi\rangle}\ -_{\neg_I} \qquad \frac{-\langle\neg\varphi\rangle}{+\langle\varphi\rangle}\ -_{\neg_E}$$

Rumfitt's Bilateralism

Rumfitt's Rules for Conjunction and Disjunction: Just Gentzen's (1935) rules, but adding negative rules exploiting the duality of conjunction and disjunction:

$$\frac{+\langle\varphi\rangle \quad +\langle\psi\rangle}{+\langle\varphi\wedge\psi\rangle}\ +\wedge\mathrm{I} \qquad \frac{+\langle\varphi\wedge\psi\rangle}{+\langle\psi\rangle}\ +_{\wedge\mathrm{E}_1} \qquad \frac{+\langle\varphi\wedge\psi\rangle}{+\langle\varphi\rangle}\ +_{\wedge\mathrm{E}_2}$$

$$\frac{-\langle\varphi\rangle}{-\langle\varphi\wedge\psi\rangle}\ -_{\wedge\mathrm{I}_2} \qquad \frac{-\langle\psi\rangle}{-\langle\varphi\wedge\psi\rangle}\ -_{\wedge\mathrm{I}_2} \qquad \frac{-\langle\varphi\wedge\psi\rangle \quad \begin{matrix}\overline{-\langle\varphi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{matrix} \quad \begin{matrix}\overline{-\langle\psi\rangle}^{\,v}\\ \vdots \\ \overline{A}\end{matrix}}{A}\ -_{\wedge\mathrm{E}}{}^{u,v}$$

$$\frac{+\langle\varphi\rangle}{+\langle\varphi\vee\psi\rangle}\ +_{\vee\mathrm{I}_2} \qquad \frac{+\langle\psi\rangle}{+\langle\varphi\vee\psi\rangle}\ +_{\vee\mathrm{I}_2} \qquad \frac{+\langle\varphi\vee\psi\rangle \quad \begin{matrix}\overline{+\langle\varphi\rangle}^{\,u}\\ \vdots \\ \overline{A}\end{matrix} \quad \begin{matrix}\overline{+\langle\psi\rangle}^{\,v}\\ \vdots \\ \overline{A}\end{matrix}}{A}\ +_{\vee\mathrm{E}}{}^{u,v}$$

$$\frac{-\langle\varphi\rangle \quad -\langle\psi\rangle}{-\langle\varphi\vee\psi\rangle}\ -_{\vee\mathrm{I}} \qquad \frac{-\langle\varphi\vee\psi\rangle}{-\langle\psi\rangle}\ -_{\vee\mathrm{E}_1} \qquad \frac{-\langle\varphi\vee\psi\rangle}{-\langle\varphi\rangle}\ -_{\vee\mathrm{E}_2}$$

Rufmitt's Bilateralism

Some Classically Validities are Provable: For example, de Morgan's laws:

$$\dfrac{\dfrac{\dfrac{\dfrac{+\langle\neg(\varphi\vee\psi)\rangle}{-\langle\varphi\vee\psi\rangle}+_{\neg_E}}{-\langle\varphi\rangle}-_{\vee_E}}{+\langle\neg\varphi\rangle}+_{\neg_I} \qquad \dfrac{\dfrac{\dfrac{+\langle\neg(\varphi\vee\psi)\rangle}{-\langle\varphi\vee\psi\rangle}+_{\neg_E}}{-\langle\psi\rangle}-_{\vee_E}}{+\langle\neg\psi\rangle}+_{\neg_I}}{+\langle\neg\varphi\wedge\neg\psi\rangle}+_{\wedge_I}$$

$$\dfrac{\dfrac{+\langle\neg(\varphi\wedge\psi)\rangle}{-\langle\varphi\wedge\psi\rangle}+_{\neg_E} \qquad \dfrac{\dfrac{\dfrac{}{-\langle\varphi\rangle}1}{+\langle\neg\varphi\rangle}+_{\neg_I}}{+\langle\neg\varphi\vee\neg\psi\rangle}+_{\vee_I} \qquad \dfrac{\dfrac{\dfrac{}{-\langle\psi\rangle}2}{+\langle\neg\psi\rangle}+_{\neg_I}}{+\langle\neg\varphi\vee\neg\psi\rangle}+_{\vee_I}}{+\langle\neg\varphi\vee\neg\psi\rangle}-_{\wedge_\mathrm{E}}{}^{1,2}$$

But Some Classically Validities are Not Provable: For example, we have neither $\vdash +\langle\varphi\vee\neg\varphi\rangle$ nor $+\langle\varphi\wedge\neg\varphi\rangle\vdash A$.

Coordination Principles

Coordination Principles: Bilateral structural rules that *coordinate* the opposite stances of affirmation and denial:

Rumfitt and Smiley: Where A and B are any signed sentences, and starring a signed sentence yields the oppositely signed sentence:

$$\dfrac{\begin{matrix} \overline{A}^{\,u} \\ \vdots \\ B \end{matrix} \qquad B^*}{A^*}\ \text{Smiley Reduc.}^u$$

Splitting Smiley Up: del Valle-Inclan (2023) notes that the following two principles are jointly equivalent to Smilean Reductio:

$$\dfrac{\begin{matrix} \overline{A}^{\,u} \\ \vdots \\ B \end{matrix} \qquad \begin{matrix} \overline{A^*}^{\,v} \\ \vdots \\ B \end{matrix}}{B}\ \text{Ex. Mid.}^{u,v} \qquad\qquad \dfrac{A \quad A^*}{B}\ \text{Explo.}$$

Note: These are *structural* rules in that they don't involve any logical connectives.

Coordination Principles

$\vdash +\langle p \lor \neg p \rangle$:

$$\dfrac{\dfrac{\dfrac{}{+\langle p \rangle}\;1}{+\langle p \lor \neg p \rangle}\;+_{\lor_I} \qquad \dfrac{\dfrac{\dfrac{}{-\langle p \rangle}\;2}{+\langle \neg p \rangle}\;+_{\neg_I}}{+\langle p \lor \neg p \rangle}\;+_{\lor_I}}{+\langle p \lor \neg p \rangle}\;\text{Ex. Mid.}^{1,2}$$

$+\langle p \land \neg p \rangle \vdash A$:

$$\dfrac{\dfrac{+\langle p \land \neg p \rangle}{+\langle p \rangle}\;+_{\land_E} \qquad \dfrac{\dfrac{+\langle p \land \neg p \rangle}{+\langle \neg p \rangle}\;+_{\land_E}}{-\langle p \rangle}\;+_{\neg_E}}{A}\;\text{Explo.}$$

Questioning Coordination

Consider the following familiar sentence:

$\boldsymbol{\lambda}$**:** λ is not true.

Question: Is λ true?

- If we say "Yes," we commit ourselves to saying "No."
- If we say "No," we commit ourselves to saying "Yes."

- **Two Plausible Responses:**
  1. Say neither "Yes" nor "No."
  2. Say both "Yes" and "No."

  - If we opt for (1), we should reject Bilateral Excluded Middle.
  - If we opt for (2), we should reject Bilateral Explosion.
  - If we think both (1) and (2) are reasonable, then perhaps we should reject both.

Four Bilateral Natural Deduction Systems

Question: What logics do we get when we drop one or both of the coordination principles, as just suggested?

Answer: Some familiar ones:

- $\mathbf{BN_{CL}}$: Operational rules + Explosion + Excluded Middle
- $\mathbf{BN_{K3}}$: Operational rules + Explosion
- $\mathbf{BN_{LP}}$: Operational rules + Excluded Middle
- $\mathbf{BN_{FDE}}$: Operational rules

Result 1: $\mathrm{BN_{FDE}}$ proves $+\langle\varphi_1\rangle, +\langle\varphi_2\rangle \ldots + \langle\varphi_n\rangle \vdash +\langle\psi\rangle$ just in case $\varphi_1, \varphi_2 \ldots \varphi_n \vDash_{\mathrm{FDE}} \psi$. Likewise for $\mathrm{BN_{LP}}$ and $\mathrm{BN_{K3}}$.

Generalized Formulation

- I just gave rules for conjunction and disjunction, but what about other connectives? The (material) conditional? The Sheffer Stroke?
- **Definition:** A *primitive binary connective* is any connective $\circ$ such that some stance $\boldsymbol{c}$ is to be taken to $\varphi \circ \psi$ just in case some stance $\boldsymbol{a}$ is taken towards φ and some stance $\boldsymbol{b}$ is taken towards ψ.
- **Assignment of Signs to All Binary Connectives:**

$$\begin{array}{cc} \wedge\!: \ \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = + & \vee\!: \ \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = - \\ |\!: \ \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = - & \downarrow\!: \ \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = + \\ \to\!: \ \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = - & \succ\!\!-\!: \ \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = + \\ -\!\!\prec\!: \ \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = + & \leftarrow\!: \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = - \end{array}$$

- **Generalized Bilateral Notation** (Simonelli 2024). Rather than formulating the rules for connectives *individually*, we can formulate harmonious and separable rules for all of them *schematically*.
  - $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ are variables ranging over $\{+, -\}$, expressing stances.
  - * is a function mapping $+$ to $-$ and $-$ to $+$

Generalized Formulation of ND System

$$\frac{}{A \vdash A}\ \text{Reflexivity} \qquad \frac{\Gamma \vdash A}{\Gamma', \Gamma \vdash A}\ \text{Weakening} \qquad \frac{\Gamma, A \vdash B \quad \Gamma \vdash A}{\Gamma \vdash B}\ \text{Cut}$$

$$\frac{\Gamma \vdash -\langle\varphi\rangle}{\Gamma \vdash +\langle\neg\varphi\rangle}\ +_{\neg_I} \qquad \frac{\Gamma \vdash +\langle\neg\varphi\rangle}{\Gamma \vdash -\langle\varphi\rangle}\ +_{\neg_E} \qquad \frac{\Gamma \vdash +\langle\varphi\rangle}{\Gamma \vdash -\langle\neg\varphi\rangle}\ -_{\neg_I} \qquad \frac{\Gamma \vdash -\langle\neg\varphi\rangle}{\Gamma \vdash +\langle\varphi\rangle}\ -_{\neg_E}$$

$$\frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}_{\circ\mathrm{I}} \qquad \frac{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_1} \qquad \frac{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle}{\Gamma \vdash \boldsymbol{b}\langle\psi\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_2}$$

$$\frac{\Gamma \vdash \boldsymbol{a}^*\langle\varphi\rangle}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_{\circ\mathrm{I}_1} \qquad \frac{\Gamma \vdash \boldsymbol{b}^*\langle\psi\rangle}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle}\ \boldsymbol{c}^*{}_{\circ\mathrm{I}_2}$$

$$\frac{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \quad \Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash A \quad \Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash A}{\Gamma \vdash A}\ \boldsymbol{c}^*{}_{\circ\mathrm{E}}$$

$$\frac{\Gamma \vdash A \quad \Gamma \vdash A^*}{\Gamma \vdash B}\ \text{Explo.} \qquad \frac{\Gamma, A \vdash B \quad \Gamma, A^* \vdash B}{\Gamma \vdash B}\ \text{Ex. Mid.}$$

Generalized Formulation of Sequent Systems

$$\frac{}{\Gamma, A \vdash A, \Delta}\ \text{Reflex.} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ \text{Weak.} \qquad \frac{\Gamma, A \vdash \Delta \quad \Gamma' \vdash A, \Delta'}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ \text{Cut}$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash \Delta}{\Gamma, +\langle\neg\varphi\rangle \vdash \Delta}\ +_{\neg L} \quad \frac{\Gamma, +\langle\varphi\rangle \vdash \Delta}{\Gamma, -\langle\neg\varphi\rangle \vdash \Delta}\ -_{\neg L} \quad \frac{\Gamma \vdash -\langle\varphi\rangle, \Delta}{\Gamma \vdash +\langle\neg\varphi\rangle, \Delta}\ +_{\neg R} \quad \frac{\Gamma \vdash +\langle\varphi\rangle, \Delta}{\Gamma \vdash -\langle\neg\varphi\rangle, \Delta}\ -_{\neg R}$$

$$\frac{\Gamma, \boldsymbol{a}\langle\varphi\rangle, \boldsymbol{b}\langle\psi\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}\langle\varphi \circ \psi\rangle \vdash \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{L}}} \qquad \frac{\Gamma \vdash \boldsymbol{a}\langle\varphi\rangle, \Delta \quad \Gamma \vdash \boldsymbol{b}\langle\psi\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}\langle\varphi \circ \psi\rangle, \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{R}}}$$

$$\frac{\Gamma, \boldsymbol{a}^*\langle\varphi\rangle \vdash \Delta \quad \Gamma, \boldsymbol{b}^*\langle\psi\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}^*\langle\varphi \circ \psi\rangle \vdash \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{L}}} \qquad \frac{\Gamma \vdash \boldsymbol{a}^*\langle\varphi\rangle, \boldsymbol{b}^*\langle\psi\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}^*\langle\varphi \circ \psi\rangle, \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{R}}}$$

$$\frac{}{\Gamma, A, A^* \vdash \Delta}\ \text{Explo.} \qquad \frac{}{\Gamma \vdash A, A^*, \Delta}\ \text{Ex. Mid.}$$

- **Four sequent systems:**
  1. $\mathbf{BS_{CL}}$**:** Operational rules + Explosion + Excluded Middle
  2. $\mathbf{BS_{K3}}$**:** Operational rules + Explosion
  3. $\mathbf{BS_{LP}}$**:** Operational rules + Excluded Middle
  4. $\mathbf{BS_{FDE}}$**:** Operational rules
- Some Proof-Theoretic Facts:
  - Cut is eliminable
  - Weakening is eliminable.
  - Reflexivity can be restricted to atomics
  - Explosion and Excluded Middle can be restricted to atomics (in systems containing them)
  - All corresponding ND rules (both operational and structural) can be derived (given Cut).

All proven *schematically*, without descending to the level of the particular binary connectives

Issues with Standard Systems

- **Issues with Standard Systems:** The rules for standard unilateral natural deduction and multiple conclusion sequent systems for the FDE family have two problems.
  - **Separability:** Rules aren't separable. Usually, De Morgan is added as a primitive rule (e.g Priest 2019, Beall 2013). Axioms for K3 and LP also generally involve logical vocabulary (at least negation).
  - **Lack of Proper Negation Rules:** The only negation rules only involving negation is double negation rules, but this doesn't suffice to characterize the distinctive behavior of negation.
- **Beall's Conclusion:** This leads Beall (2017), a proponent of FDE, to conclude that "logic itself" says nothing about negation other than its "interaction with other logical connectives (e.g., $\neg\wedge$, $\neg\vee$ etc.)," (15), ultimately maintaining that "there is no logical negation."
- **However:** In all systems here, the rules for all the connectives are *separable*, and the rules of negation are just those of bilateral logic, as put forward by Smiley and Rumfitt. Negation is an operator that flip-flops between affirmation and denial.

Issues with Non-Standard Sequent Systems

- In response to the technical issues of standard systems, non-standard (4-signed, 4-sided, or dual 2-sided) sequent systems have been proposed (e.g. Shapiro (2016), Fjellstad (2016), Wintein (2016)).
  - The sequent systems proposed here are technically quite similar.
- **But:** These existing non-standard sequent systems lack an intuitive interpretation.
  - Wintein (2016) proposes "quadrilateralism," with two notions of assertions and two notions of denial.
- The sequent systems put forward here are simply bilateral, with a single notion of assertion and a single notion of denial, and are straightforwardly intuitively intelligible.
- **Moreover:** If you don't like multiple conclusions, each sequent system has a corresponding natural deduction system!

Towards a Bilateral Semantics

- These systems are sound and complete with respect to *uni*lateral validity, in the sense defined above:
  - $\mathrm{BS}_{\mathrm{FDE}}$ proves $+\langle\varphi_1\rangle, +\langle\varphi_2\rangle \ldots + \langle\varphi_n\rangle \vdash +\langle\psi_1\rangle, +\langle\psi_2\rangle \ldots + \langle\psi_n\rangle$ just in case $\varphi_1, \varphi_2 \ldots \varphi_n \vDash_{\mathrm{FDE}} \psi_1, \psi_2 \ldots \psi_n$. Likewise for $\mathrm{BS}_{\mathrm{LP}}$ and $\mathrm{BS}_{\mathrm{K3}}$.
- But this only considers a fragment of the *bi*lateral consequence relation generated by each logic.
- It will prove fruitful to define a notion of *bilateral* validity with respect to which they are sound and complete.

Generalized Bilateral Semantics

- **Correctness (Informal Characterization):** Affirming some sentence φ is correct just in case φ is true, and denying φ is correct just in case φ is false.
  - Some sentences may be neither true nor false, so neither correct to affirm nor correct to deny.
  - Some sentences may be both true and false, so both correct to affirm and correct to deny.
- **Four Possible Valuations:** $\varnothing, \{1\}, \{0\}, \{1,0\}$
  - φ is just true (correct to affirm, incorrect to deny): $v(\varphi) = \{1\}$
  - φ is just false (incorrect to affirm, correct to deny): $v(\varphi) = \{0\}$
  - φ is neither true nor false (incorrect to affirm, incorrect to deny): $v(\varphi) = \varnothing$
  - φ is both true and false (correct to affirm, correct to deny): $v(\varphi) = \{1,0\}$

Generalized Bilateral Semantics

- **Correctness Function:** The *correctness function* $[]$ is a function from $\{+, -\}$ to $\{1, 0\}$ mapping $+$ to 1 and $-$ to 0.
- **4-valued Correctness:** Taking some stance $\boldsymbol{a}$ towards some sentence φ, $\boldsymbol{a}\langle\varphi\rangle$, is *correct*, relative to some valuation v, just in case $[\boldsymbol{a}] \in v(\varphi)$.
  - Read $[\boldsymbol{a}]$ as "a truth value that would make stance $\boldsymbol{a}$ correct."
- **Four Valued Valuations:** A *four-valued valuation* v is any function from $\mathcal{L} \to \{\varnothing, \{1\}, \{0\}, \{1, 0\}\}$ that assigns an element of this set to each atomic sentence p and recursively assigns values to complex sentences as follows:

$$v(\neg\varphi) \ni \begin{cases} 1, & \text{if } 0 \in v(\varphi) \\ 0, & \text{if } 1 \in v(\varphi) \end{cases}$$

$$v(\varphi \circ \psi) \ni \begin{cases} [\boldsymbol{c}], & \text{if } [\boldsymbol{a}] \in v(\varphi) \text{ and } [\boldsymbol{b}] \in v(\psi) \\ [\boldsymbol{c}^*], & \text{if } [\boldsymbol{a}^*] \in v(\varphi) \text{ or } [\boldsymbol{b}^*] \in v(\psi) \end{cases}$$

Example

- **Semantic Schema for Binary Connectives:**

$$v(\varphi \circ \psi) \ni \begin{cases} [\boldsymbol{c}], & \text{if } [\boldsymbol{a}] \in v(\varphi) \text{ and } [\boldsymbol{b}] \in v(\psi) \\ [\boldsymbol{c}^*], & \text{if } [\boldsymbol{a}^*] \in v(\varphi) \text{ or } [\boldsymbol{b}^*] \in v(\psi) \end{cases}$$

- **Recall:** The following assignments of signs to connectives:

$$\begin{array}{rl} \wedge\colon \ \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = + & \qquad \vee\colon \ \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = - \\ |\colon \ \boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = - & \qquad \downarrow\colon \ \boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = + \\ \rightarrow\colon \ \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = - & \qquad \succ\!\!-\colon \ \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = + \\ -\!\!\prec\colon \ \boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = + & \qquad \leftarrow\colon \boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = - \end{array}$$

So: We have the following clause for $|$, the Sheffer Stroke:

$$v(\varphi \mid \psi) \ni \begin{cases} 0, & \text{if } 1 \in v(\varphi) \text{ and } 1 \in v(\psi) \\ 1, & \text{if } 0 \in v(\varphi) \text{ or } 0 \in v(\psi) \end{cases}$$

Which gives us the following truth table:

$\mid$	$\{1\}$	$\{1, 0\}$	$\varnothing$	$\{0\}$
$\{1\}$	$\{0\}$	$\{1, 0\}$	$\varnothing$	$\{1\}$
$\{1, 0\}$	$\{1, 0\}$	$\{1, 0\}$	$\{1\}$	$\{1\}$
$\varnothing$	$\varnothing$	$\{1\}$	$\varnothing$	$\{1\}$
$\{0\}$	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

It is easy to check, given this truth table, that $\varphi \mid \psi$ is equivalent to $\neg(\varphi \wedge \psi)$.

Likewise, we get the truth-tables for all of the other connectives.

Bilateral Validity

Admissible Valuations:

1. **CL:** All valuations $\mathcal{A} \to \{\{1\}, \{0\}\}$
2. **LP:** All valuations $\mathcal{A} \to \{\{1\}, \{0\}, \{1, 0\}\}$
3. **K3:** All valuations $\mathcal{A} \to \{\varnothing, \{1\}, \{0\}\}$
4. **FDE:** All valuations $\mathcal{A} \to \{\varnothing, \{1\}, \{0\}, \{1, 0\}\}$

- **Bilateral Validity:** An argument of the form $\Gamma \vdash \Delta$ is *bilaterally valid*, relative to a set of admissible valuations V, $\Gamma \vDash_{B_V} \Delta$, just in case there is no $v \in V$ such that all of the stances in Γ are correct and all of the stances in Δ are incorrect.
- **Main Result:** All proof systems are sound and complete with respect to bilateral validity.
  - *Proven schematically, for any systems containing any selection of connectives.*

Unilateral and Bilateral Validity

- **Unilateral Validity** An argument of the form $X \vdash Y$ is *unilaterally valid*, relative to a set of admissible valuations V, $X \vDash_{U_V} Y$ just in case there is no $v \in V$ such $1 \in v(\varphi)$ for all $\varphi \in X$ and $1 \notin v(\psi)$ for all $\psi \in Y$.
- **Some Facts:** Where $+\langle X\rangle$ as shorthand for $+\langle\varphi_1\rangle, +\langle\varphi_2\rangle \ldots + \langle\varphi_n\rangle$ for all $\varphi \in X$:
  - $+\langle X\rangle \vDash_{B_V} +\langle Y\rangle$ just in case $X \vDash_{U_V} Y$.
  - $+\langle X\rangle, -\langle Y\rangle \vDash_{B_{\mathrm{K3}}}$ just in case $X \vDash_{U_{\mathrm{CL}}} Y$
  - $\vDash_{B_{\mathrm{LP}}} -\langle X\rangle, +\langle Y\rangle$ just in case $X \vDash_{U_{\mathrm{CL}}} Y$

Philosophical Upshot

Moving from *uni*lateral to *bi*lateral consequence has important upshots for how we think about the debate between "substructural" and "subclassical" approaches to paradox.

Restall/Ripley Bilateralism

- **Bilateral Interpretation of Multiple Conclusion Sequents:** A multiple conclusion unilateral sequent $X \vdash Y$ can be understood as saying that asserting everything in X along with denying everything in Y is "incoherent" or "out of bounds," (Restall 2005, Ripley 2013).
  - In other words, there's no way for it to be correct to assert everything in X and deny everything in Y.
  - In our system, a solely left-sided bilateral sequent of the form $\Gamma \vdash$ says that it can't be the case that all of the stances in Γ are correct.
- **Making Restall/Ripley Bilateralism Explicit:** To translate a multiple conclusion unilateral sequent $X \vdash Y$ to a solely left-sided bilateral sequent $\Gamma \vdash$, just let

$$\Gamma = \{+\langle\varphi\rangle \mid \varphi \in X\} \cup \{-\langle\psi\rangle \mid \psi \in Y\}.$$

Classical Sequent Calculus

$$\frac{}{X, \varphi \vdash \varphi, Y}$$

$$\frac{X \vdash \varphi, Y}{X, \neg\varphi \vdash Y}\ \mathrm{L}_{\neg} \qquad \frac{X, \varphi \vdash Y}{X \vdash \neg\varphi, Y}\ \mathrm{R}_{\neg}$$

$$\frac{X, \varphi, \psi \vdash Y}{X, \varphi \wedge \psi \vdash Y}\ \mathrm{L}_{\wedge} \qquad \frac{X \vdash \varphi, Y \quad X \vdash \psi, Y}{X \vdash \varphi \wedge \psi, Y}\ \mathrm{R}_{\wedge}$$

$$\frac{X, \varphi \vdash Y \quad X, \psi \vdash Y}{X, \varphi \vee \psi \vdash Y}\ \mathrm{L}_{\vee} \qquad \frac{X \vdash \varphi, \psi, Y}{X \vdash \varphi \vee \psi, Y}\ \mathrm{R}_{\vee}$$

$$\frac{X \vdash \varphi, Y \quad X, \psi \vdash Y}{X, \varphi \to \psi \vdash Y}\ \mathrm{L}_{\to} \qquad \frac{X, \varphi \vdash \psi, Y}{X \vdash \varphi \to \psi, Y}\ \mathrm{R}_{\to}$$

Solely Left-Sided Fragment of BSK3 or BSCL:

$$\frac{}{\Gamma, A, A^* \vdash}\ \text{Explo.}$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash}{\Gamma, +\langle\neg\varphi\rangle \vdash}\ +_\neg \qquad \frac{\Gamma, +\langle\varphi\rangle \vdash}{\Gamma, -\langle\neg\varphi\rangle \vdash}\ -_\neg$$

$$\frac{\Gamma, +\langle\varphi\rangle, +\langle\psi\rangle \vdash}{\Gamma, +\langle\varphi \wedge \psi\rangle \vdash}\ +_\wedge \qquad \frac{\Gamma, -\langle\varphi\rangle \vdash \quad \Gamma, -\langle\psi\rangle \vdash}{\Gamma, -\langle\varphi \wedge \psi\rangle \vdash}\ -_\wedge$$

$$\frac{\Gamma, +\langle\varphi\rangle \vdash \quad \Gamma, +\langle\psi\rangle \vdash}{\Gamma, +\langle\varphi \vee \psi\rangle \vdash}\ +_\vee \qquad \frac{\Gamma, -\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma, -\langle\varphi \vee \psi\rangle \vdash}\ -_\vee$$

$$\frac{\Gamma, -\langle\varphi\rangle \vdash \quad \Gamma, +\langle\psi\rangle \vdash}{\Gamma, +\langle\varphi \to \psi\rangle \vdash}\ +_\to \qquad \frac{\Gamma, +\langle\varphi\rangle, -\langle\psi\rangle \vdash}{\Gamma, -\langle\varphi \to \psi\rangle \vdash}\ -_\to$$

Question: Should we think of this as the solely left-sided fragment of BSK3 or BSCL?

Ripley's Bilateralism, Made Explicit:

- **Ripley's Solution to the Liar:** Reject (unilteral) Cut:
  - In standard unilateral notation:

$$\frac{X, \varphi \vdash Y \quad X \vdash \varphi, Y}{X \vdash Y}\ \text{Unilateral Cut}$$

  - More perspicuously, in explicitly bilateral notation:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash \quad \Gamma, -\langle\varphi\rangle \vdash}{\Gamma \vdash}\ \text{Extensibility}$$

  - Extensibility is sound in BCL, but unsound in BK3. Why? Because it's just the solely left-sided version of Excluded Middle:

$$\frac{\Gamma, +\langle\varphi\rangle \vdash \Delta \quad \Gamma, -\langle\varphi\rangle \vdash \Delta}{\Gamma \vdash \Delta}\ \text{Excluded Middle}$$

    - ★ The notion of bilateral validity that we've defined for the solely left-sided fragment of BK3 *just is* that of unilateral ST.

Extending Ripley's Approach

- By translating into explicitly bilateral notation, we can extend Ripley's solution to the other side of the turnstile, so that we don't just talk about *incoherence* of sets of assertions and denials, but also *consequence relations*: which sets of stances *commit* one to other stances.
- **Two Coordination Principles worth Highlighting:**

$$\frac{\Gamma, A \vdash}{\Gamma \vdash A^*}\ \text{Out}$$

A proponent of Ripley's approach should reject this.

$$\frac{\Gamma \vdash A,}{\Gamma, A^* \vdash}\ \text{In}$$

A proponent of Ripley's approach should accept this.

- **Fact:** Bilateral K3 rejects Out and accepts In.
- **My Claim:** The bilateral logic that Ripley is implicitly working with really is that of Bilateral K3.

Upshot for Subclassical vs. Substructural Approaches:

- **Ripley's Claim of Classicality:** "The core of the advantages of the ST approach over K3T- and LPT-based approaches" is that it enables one to have a transparent truth-predicate while retaining all of classical logic.
  - **In one sense, this is true:** All of *unilateral* classical logic is retained in this solution.
  - **In another sense, this is not true:** All of *bilateral* classical logic is not retained. For instance, even though Bilateral K3 validates $-\langle \varphi \vee \neg\varphi \rangle \vdash$, it does not validate $\vdash +\langle \varphi \vee \neg\varphi \rangle$.
- **Ripley's Claim of Substructurality:** Ripley advertises his approach as requiring us "to let go of cut, and with it transitivity of consequence," (2013, 155).
  - **In one sense, this is true:** We must let go of *unilateral* Cut (i.e. Extensibility).
  - **In a deeper sense, this is not true:** The consequence relation of Bilateral K3 is *completely transitive.* Insofar as this is a faithful extension of Ripley's solution, Ripley's approach does not require us to give up Transitivity for *bilateral* logic.

Summing Up:

- Adopting a bilateral approach enables one to put formulate technically elegant and intuitive proof systems for the logics in the FDE family.
  - And to do so *schematically*.
- This also has important philosophical consequences for how we think about "substructural" vs. "subclassical" approaches to paradox.

Thanks!

The Paper

Available at ryansimonelli.com/papers or scan here:

References

Slides available at ryansimonelli.com/papers (last section)

Boyle, Mathew. 2012. "Essentially Rational Animals." In *Rethinking Epistemology*, ed. G. Abel and J. Conant, 395-427. Berlin: de Gruyter.

Brandom, Robert. 1994. *Making It Explicit.* Cambridge, MA: Harvard University Press.

Gentzen, Gerhard. 1935. "Investigations into Logical Deduction," in *The Collected Papers of Gerhard Gentzen*, ed. M. Szabo, 68-131. Amsterdam: North- Holland. 1969.

Haugeland, John. 1998. "Understanding Natural Language." In *Having Thought*, 47-61. Cambridge, MA: Harvard University Press.

Jackson, Frank. 1982. "Epiphenomenal Qualia." *Philosophical Quarterly* 32, no. 127: 127-136.

Sellars, Wilfrid. 1956. " Empiricism and the Philosophy of Mind." *Minnesota Studies in the Philosophy of Science* 1:253-329.

Simonelli, Ryan. 2023. "How to Be a Hyper-Inferentialist." *Synthese* 202, no. 163.