“Yes,” “No,” Neither, and Both

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Abstract

When faced with the question of whether to affirm or deny a paradoxical sentence such as the liar, it seems that there are two plausible responses: neither affirming it nor denying it, or both affirming it and denying it. In this paper, I make this thought concrete by formulating bilateral proof systems (of both the natural deduction and sequent calculus variety) for the logics in the FDE family: K3, LP, and FDE. The different logics are simply the result of different choices of “coordination principles,” bilateral structural rules which coordinate the opposite stances of affirmation and denial. I show that conceiving of these logics in bilateral terms has important philosophical consequences, most notably for the debate between “non-classical” and “substructural” approaches to paradox. In particular, I show how adopting Bilateral K3 enables one to endorse the “non-transitive” solution to paradox, as developed by Ripley, while maintaining that logical consequence (understood *bi*laterally) is transitive.

Key Words: Bilateralism; Paradox; Non-Classical Logic; FDE; LP; K3; ST

0 Introduction

Bilateral proof systems provide rules both for affirming and denying sentences. With some recent exceptions, bilateral systems have generally been proposed in the context of classical logic with the aim of providing harmonious rules for the classical connectives.[1] In classical systems, affirmation and denial are taken to be exhaustive and exclusive such that, for every sentence A, taking exactly one of these two opposite stances is correct; it can never be that neither are correct nor can it ever be that both are correct. This is formally implemented by a bilateral logic's inclusion of “coordination principles,” bilateral structural rules that formally codify the relation between affirmation and denial. In this paper, I argue that, when A is a paradoxical sentence such as the liar, it is plausibly reasonable for one to neither affirm nor deny A, or, alternatively, for one to both affirm and deny A.

[1]Most notably, see Rumfitt [37]. Recent exceptions have been largely in the context of intuitionistic logic (e.g. [47], [3]), but see also [48]. It's worth noting, however, that, in this context “bilateralism” is not understood in terms of opposite speech acts of affirmation and denial, but, rather, in terms of the verification and falsification of sentences.

This suggests exploring bilateral logics in which the coordination principles that enforce classicality are weakened or dropped entirely. I show that, by doing this, one arrives at technically elegant and straightforwardly intuitive systems for the non-classical logics in the FDE family: K3, LP, and FDE.[2] I propose both bilateral natural deduction and bilateral sequent systems for each of these logics, and I argue that investigating the distinctively *bi*lateral consequence relations of these logics has some important philosophical consequences.

The most notable case, on which I focus here, is the system I call "Bilateral K3," so-called because it contains K3's unilateral consequence relation as its solely positive fragment. Notably, however, it also contains all of unilateral classical logic's consequence relation in its solely left-sided fragment. As it turns out, the solely left-sided fragment of Bilateral K3 turns out to just be a notational variant of the logic ST, appealed to by Ripley and others [32] [7] [33] [34] in response to the liar paradox, with Restall's [30] bilateral interpretation to which Ripley [33] appeals made explicit in the bilateral notation itself. Formulated in this bilateral setting, it becomes clear that Ripley's rejection of unilateral Cut is not a rejection of *transitivity*, but, rather, the rejection of a specific kind of bilateral *excluded middle* principle. This result is significant in the context of the debate between "non-classical" and "substructural" approaches to semantic paradox. Ripley's approach is advertised as a substructural approach that enables us to maintain classicality. Formulated in the bilateral system put forward here, however, Ripley's approach ends up looking much less classical than it looks on his own presentation of it; Bilateral K3 contains all unilateral classical, but (in a sense that I will explain) it does so only on the *left side* of the turnstile. The consequence relation *across* the turnstile is not classical. So, whether or not Ripley's account is classical or not depends on whether we're talking about *unilateral* classical logic or *bilateral* classical logic. Moreover, if we're talking about the bilateral consequence relation, Ripley's account can be seen as fully structural. In particular, the structural rule of *Cut* or *Transitivity*, understood as pertaining to the *bilateral* consequence relation, can be maintained, even with the vocabulary of a truth predicate and liar sentence added.

The paper is structured as follows. In Section 1, I lay out Rumfitt's bilateral system for classical logic, with a few minor (and well-motivated) tweaks. In Section 2, I motivate the *neither* or *both* responses to the liar paradox. In Section 3, I show

[2] K3 is Kripke's [23] "Logic of Truth" (see also Kremer [22]). LP is Preist's [26] "Logic of Paradox," first proposed by Asenjo [2]. FDE is Anderson and Belnaps's [1] logic of "First Degree Entailment." For an introductory overview of these logics, see Beall, Glanzberg and Ripley [5], Chapter 5. These logics will be explained in Section 3.

how, by dropping one or both of the two coordination principles that rule out such options from Rumfitt's system, one arrives at bilateral natural deduction systems for K3, LP, and FDE. I also extend the conception of bilateral consequence to multiple conclusion arguments and put forward bilateral sequent systems for the FDE family. In Section 4, I return to the two responses to the liar initially suggested in Section 2, and explicate them formally with the use of these new bilateral logics. In Section 5, I consider the implications of this approach for Ripley's "non-transitive" solution to the liar paradox. The Appendix provides the technical results left out of the body of the paper, deploying the generalized approach to bilateralism proposed by Simonelli [40] [41].

1 Bilateral Classical Logic

A bilateral proof system of the sort proposed by Smiley [42] and Rumfitt [37] provides rules both for affirming and denying sentences. In such a system, formulas are positively or negatively signed, expressing affirmation or denial. Where A is any sentence, $+\langle A\rangle$ expresses the affirmation of A, whereas $-\langle A\rangle$ expresses the denial of A. Unlike a negation operator, these signs are neither embeddable nor iterable; there must always be exactly one sign and it must always be prefixed to a whole sentence. So, for instance, although both $+\langle p \wedge \neg q\rangle$ and $-\langle \neg p\rangle$ are well-formed, neither $+\langle p \wedge -\langle q\rangle\rangle$ nor $-\langle -\langle p\rangle\rangle$ are well-formed.

The most well-known bilateral system is the bilateral natural deduction system proposed by Rumfitt [37] in response to Dummett's [10] criticism of classical natural deduction having unharmonious negation rules.[3] Rumfitt shows that, by going bilateral, one is able to arrive at a perfectly harmonious system for classical logic. In Rumfitt's bilateral system, the negation rules are the following:

$$\frac{-\langle A\rangle}{+\langle \neg A\rangle}\ +_{\neg_I} \qquad\qquad \frac{+\langle \neg A\rangle}{-\langle A\rangle}\ +_{\neg_E}$$

$$\frac{+\langle A\rangle}{-\langle \neg A\rangle}\ -_{\neg_I} \qquad\qquad \frac{-\langle \neg A\rangle}{+\langle A\rangle}\ -_{\neg_E}$$

Reading the horizontal line as expressing commitment, as suggested by Incurvati and Schlöder [19] [18], the rules for affirming a negation say that denying A commits

[3]There are two systems stated in Rumfitt's article: the "more compact" system proposed by Smiley and the one Rumfitt himself proposes. The system considered here is the latter. See [24] for a discussion of these two systems and an explanation of why the latter is preferable in the context of Rumfitt's project.

one to affirming $\neg A$, and affirming $\neg A$ commits one to denying A. Likewise, the rules for denying a negation say that affirming A commits one to denying $\neg A$, and denying $\neg A$ commits one to affirming A. These rules are clearly harmonious, and double negation introduction and elimination are directly proven through two applications of the I-rules and E-rules respectively.

The rules for conjunction and disjunction proposed by Rumfitt are just the standard conjunction and disjunction rules from Gentzen [15], taken as the positive rules for each connective, with each connective supplemented with rules of the form of the other connective for its negative rules. Where φ is any signed formula, the rules are the following :

$$\frac{+\langle A\rangle \quad +\langle B\rangle}{+\langle A\wedge B\rangle}\ {+\wedge_{\mathrm{I}}} \qquad \frac{+\langle A\wedge B\rangle}{+\langle B\rangle}\ {+\wedge_{\mathrm{E}_1}} \qquad \frac{+\langle A\wedge B\rangle}{+\langle A\rangle}\ {+\wedge_{\mathrm{E}_2}}$$

$$\frac{-\langle A\rangle}{-\langle A\wedge B\rangle}\ {-\wedge\mathrm{I}_1} \qquad \frac{-\langle B\rangle}{-\langle A\wedge B\rangle}\ {-\wedge\mathrm{I}_2} \qquad \frac{-\langle A\wedge B\rangle \quad \begin{array}{c}\overline{-\langle A\rangle}^{\,u}\\ \vdots\\ \overline{\varphi}\end{array} \quad \begin{array}{c}\overline{-\langle B\rangle}^{\,v}\\ \vdots\\ \overline{\varphi}\end{array}}{\varphi}\ {-\wedge_{\mathrm{E}}}^{u,v}$$

$$\frac{+\langle A\rangle}{+\langle A\vee B\rangle}\ {+\vee\mathrm{I}_1} \qquad \frac{+\langle B\rangle}{+\langle A\vee B\rangle}\ {+\vee\mathrm{I}_2} \qquad \frac{+\langle A\vee B\rangle \quad \begin{array}{c}\overline{+\langle A\rangle}^{\,u}\\ \vdots\\ \overline{\varphi}\end{array} \quad \begin{array}{c}\overline{+\langle B\rangle}^{\,v}\\ \vdots\\ \overline{\varphi}\end{array}}{\varphi}\ {+\vee_{\mathrm{E}}}^{u,v}$$

$$\frac{-\langle A\rangle \quad -\langle B\rangle}{-\langle A\vee B\rangle}\ {-\vee_{\mathrm{I}}} \qquad \frac{-\langle A\vee B\rangle}{-\langle A\rangle}\ {-\vee_{\mathrm{E}_1}} \qquad \frac{-\langle A\vee B\rangle}{-\langle B\rangle}\ {-\vee_{\mathrm{E}_2}}$$

These rules very clearly capture the duality of conjunction and disjunction. In general, in a bilateral system, where positive and negative rules have been provided for one connective, rules for its dual can be reached simply by taking the set of rules with all of the signs reversed.

Now, the rules for the material conditional that Rumfitt provides are different than the rules for conjunction and disjunction. The *negative* rules for the conditional are of the same form as the positive conjunction rule and the negative disjunction rules:

$$\frac{+\langle A\rangle \quad -\langle B\rangle}{-\langle A\supset B\rangle}\ {-\supset_{\mathrm{I}}} \qquad \frac{-\langle A\supset B\rangle}{+\langle A\rangle}\ {-\supset_{\mathrm{E}_1}} \qquad \frac{-\langle A\supset B\rangle}{-\langle B\rangle}\ {-\supset_{\mathrm{E}_2}}$$

As Rumfitt says, these negative rules for the conditional "bring out very clearly its materiality" (803), really showing that one is to deny a conditional in just the case that one affirms its antecedent and denies its consequent. Insofar as these negative rules for the conditional have the same "conjunctive" form as the positive conjunction and negative disjunction rules, it is reasonable to assign positive rules for the conditional that have the same "disjunctive" form as the negative conjunction and positive disjunction rules. That is, rather than the rules Rumfitt assigns, which are the same as Gentzen's rules, it is reasonable to assign the following rules:[4]

$$\frac{-\langle A\rangle}{+\langle A\supset B\rangle}\,{+}{\supset}\mathrm{I}_1 \qquad\qquad \frac{+\langle B\rangle}{+\langle A\supset B\rangle}\,{+}{\supset}\mathrm{I}_2 \qquad\qquad \dfrac{+\langle A\supset B\rangle \qquad \begin{matrix}\overline{-\langle A\rangle}^{\;u}\\ \vdots \\ \overline{\varphi}\end{matrix} \qquad \begin{matrix}\overline{+\langle B\rangle}^{\;v}\\ \vdots\\ \overline{\varphi}\end{matrix}}{\varphi}\,{+}{\supset}_{\mathrm{E}}{}^{u,v}$$

These conditional rules, even more clearly than Rumfitt's, bring out the materiality of the conditional.

Several important classically valid inferences follow from these rules. For instance, it is easy to see that all of De Morgan's laws can be proven, given these rules. However, many classical validities are not provable with just these rules. For instance, we have neither $\vdash +\langle A\vee\neg A\rangle$ nor $+\langle A\wedge\neg A\rangle\vdash\varphi$. To arrive at a classical system, bilateralists like Smiley and Rumfitt add what are called "coordination principles," bilateral structural rules which "coordinate" the opposite stances of affirmation and denial. The key coordination principle for classical logic put forward by Smiley is what Rumfitt calls *Smileian Reductio*:[5]

$$\dfrac{\begin{matrix}\overline{\varphi}^{\;u}\\ \vdots\\ \psi \qquad \psi^*\end{matrix}}{\varphi^*}\,\text{Smiley Reduc.}^{u}$$

[4]In a unilateral context, rules of this form will suffer from a lack of separability, as formulating them will essentially involve negation; however, in this bilateral context, there is no such issue.

[5]This is Smiley's original formulation. More recent formulations of coordination principles for classical logic (e.g.[19], [9], [8], [18]) follow Rumfitt [37, 804] in splitting up it up into the following two principles (confusingly, they call the latter principle "Smileian Reductio," when the original coinage of term by Rumfitt refers to the combined principle, which does not feature $\bot$):

$$\frac{\varphi \quad \varphi^*}{\bot} \qquad\qquad\qquad \dfrac{\begin{matrix}\overline{\varphi}^{\;u}\\ \vdots\\ \bot\end{matrix}}{\varphi^*}\,{}^{u}$$

In the context of the framework to be developed here (in Section 3), the former can be identified with the instance of (meta-inferential) Explosion where Δ is empty, and the latter can be identified with the instance of Out where Δ is empty.

Here, φ and ψ are any signed sentences, and starring a signed sentence yields the oppositely signed sentence. So, this principle says that if, given the supposition of some stance φ, one can conclude two opposite stances, ψ and ψ^*, then one can conclude φ^*, the opposite of φ. As del Valle-Inclan [8] has noted, Smileian Reductio is inter-derivable with the following two principles, *Bilateral Excluded Middle* and *Bilateral Explosion*:[6]

$$\dfrac{\begin{array}{cc}\overline{\varphi}^{\,u} & \overline{\varphi^*}^{\,v} \\ \vdots & \vdots \\ \psi & \psi\end{array}}{\psi}\ \text{Ex. Mid.}^{u,v} \qquad\qquad \dfrac{\varphi \quad \varphi^*}{\psi}\ \text{Explo.}$$

Splitting Smileian Reductio into Excluded Middle and Explosion in this way is perhaps a bit more perspicuous in showing the assumptions built into Smileian Reductio, since it's easy to see how imposing these two principles amounts to building in the assumptions of exhaustivity and exclusivity of the correctness of affirmation and denial. If we think of a proof as valid just in case it never takes us from correct stances to incorrect ones, then excluded Middle can be seen as building in the assumption that *at least one* of the opposite stances φ and φ^* must always be correct. After all, if ψ follows from φ and ψ follows from φ^*, then, given that at least one of these two opposite stances must be correct, we can conclude that ψ is correct. Explosion, on the other hand, can be understood as building in the assumption that *at most one* of the opposite stances φ and φ^* can ever be correct. After all, insofar as it can never be the case that both φ and φ^* are correct, inferring any stance ψ from these stances will never take you from correct stances to an incorrect one.

[6]The derivation of Smileian Reductio from Excluded Middle and Explosion goes as follows:

$$\dfrac{\dfrac{\begin{array}{cc}\overline{\varphi}^{\,1} & \\ \vdots & \\ \psi & \psi^*\end{array}}{\varphi^*}\ \text{Explo.} \qquad \overline{\varphi^*}^{\,2}}{\varphi^*}\ \text{Ex. Mid.}^{1,2}$$

The derivations of Excluded Middle and Explosion from Smileian Reductio go as follows:

$$\dfrac{\dfrac{\begin{array}{cc}\overline{\varphi}^{\,1} & \\ \vdots & \\ \psi & \overline{\psi^*}^{\,3}\end{array}}{\varphi^*}\ \text{Smiley Reduc.}^{1} \qquad \dfrac{\begin{array}{cc}\overline{\varphi^*}^{\,2} & \\ \vdots & \\ \psi & \overline{\psi^*}^{\,3}\end{array}}{\varphi}\ \text{Smiley Reduc.}^{2}}{\psi}\ \text{Smiley Reduc.}^{3} \qquad\qquad \dfrac{\varphi \quad \varphi^*}{\psi}\ \text{Smiley Reduc.}^{0}$$

2 Two Responses to the Liar

While the assumptions of the exhaustivity and exclusivity of affirmation and denial are plausible in the restricted context of classical propositional logic, we might wonder how they fare when we extend bilateral logic to contexts where classical logic has been called into question. Perhaps the most famous context is in debates surrounding the following sentence:

> **ℓ:** ℓ is not true.

Is ℓ true? Suppose we say "Yes," affirming that ℓ is true. If it's true, then what it says is true, but what it says is that it is not true, and so, if that's true, then it's not true. It seems, then, that if we say "Yes," affirming that ℓ is true, we're committed to saying "No," denying that ℓ is true. Suppose, then, that we say "No," denying that ℓ is true. If it's not true, then, given that what it says is that it's not true, it says something true, and so it's true. So, if we say "Yes," in response to the question of whether ℓ is true, we're committed to saying "No," and if we say "No," we're committed to saying "Yes." Thus, if we say either "Yes" or "No," then, we're committed to saying both "Yes" and "No." What, then, should we say in response to the question of whether ℓ is true? There are, I think, two plausible responses.

The first response is to say nothing. That is, in response to the question of whether ℓ is true, we say neither "Yes" nor "No." This seems like what we ought to do in response to the question insofar as we don't want to commit ourselves to saying both "Yes" and "No." As we've just seen, if we say either, we commit ourselves to saying both. So, we might say neither.[7] If we take this line in response to the liar, then we should maintain that affirming the liar is incorrect, but that this doesn't mean that denying the liar is correct. On the contrary, denying the liar is incorrect too.

The second response is to say something. Now, as we've seen, insofar as we say "Yes" in response to the question of whether ℓ is true, we are committed to saying "No" to this question, and insofar as we say "No" to this question, we are committed to saying "Yes." Accordingly, insofar as we say something, the only thing we can coherently say is both "Yes" and "No." If we take this line in response

[7] Crucially, saying neither "Yes" nor "No" is distinct from saying "Neither 'Yes' nor 'No'." Saying that latter thing is tantamount to affirming $\neg(\ell \vee \neg\ell)$, and, doing that (as we'll see officially shortly) will commit one to both affirming ℓ and denying ℓ, precisely the thing one wants to avoid in being silent in response to the question of whether ℓ. So, if one wishes to be silent in response to the question of ℓ, one should also be silent in response to the question of $\ell \vee \neg\ell$, as taking either positive or negative stance in response to this question will commit one to taking both in response to the question of ℓ.

to the liar, then we should maintain that affirming the liar is correct, and denying it is also correct, but that doesn't mean, for instance, that affirming that the moon is made of cheese is correct. On the contrary, it's incorrect to affirm that the moon is made of cheese.[8]

I do not know which of these two responses is to be preferred, but they both seem *prima facie* plausible to me. The proponent of the first response denies that affirmation and denial are *exhaustive*: there are some sentences A (for instance, ℓ) such that neither affirming A nor denying A is correct. Accordingly, one cannot assume that one must be correct, as one implicitly does in using Bilateral Excluded Middle. The proponent of the second response denies that affirmation and denial are *exclusive*: there are some sentences A (for instance, ℓ) such that both affirming A and denying A are correct. Accordingly, one cannot assume that they can never both be correct, as one implicitly does in using Explosion. Given that rejecting either of these coordination principles can be motivated on these grounds, it is natural to wonder what consequence relations one gets if excludes one or both of these principles from a bilateral system. It turns out, perhaps unsurprisingly, that we get the bilateral versions of some familiar logics.

3 Bilateral Systems for the FDE Family

Let us officially introduce talk of "truth" and "falsity" on the stipulation that a sentence is correct to affirm just in case it is true and correct to deny just in case it is false.[9] Then the first response amounts to admitting *truth-value gaps*, whereas the second response amounts to admitting *truth-value gluts*. To spell this out officially, then, let us suppose that there are four possible valuations a sentence might have: $\varnothing$ (neither true nor false), $\{1\}$ (just true), $\{0\}$ (just false), or $\{1, 0\}$ (both true and false). We will define the *correctness conditions* of affirmations and denials as follows:

Correctness: Affirming A is *correct*, relative to some valuation v, just in

[8]It's worth note that this line, while recognizably dialetheic, diverges from Priest's [27, 103-106] proposal for understanding the relation between denial and negation (see also Smiley and Priest [43]). Notably, Priest attempts to *sever* the tight inferential connection between denial and negation maintained by bilateralists, maintaining that one should assert "It's not the case that ℓ" but that one should not deny that ℓ. Restall [31] extends this severance of denial and negation to the gap theorist. However, insofar as the bilateralist account of negation is an attractive, I think there is good reason to try to avoid this severance if we can. This paper shows how we can.

[9]This fact might be understood in the context of an "inferential deflationist" account of truth, according to which the use of "true" with respect to a sentence functions to commit one to asserting it, and "false" with respect to a sentence functions to commit one to denying it. See [18] for a recent defense of this approach in a bilateralist framework.

case $1 \in v(A)$. Denying A is *correct*, relative to v, just in case $0 \in v(A)$.

Now, let a *four-valued valuation v* be any function from $\mathcal{L} \to \{\varnothing, \{1\}, \{0\}, \{1, 0\}\}$ that assigns an element of this set to each atomic sentence p and recursively assigns values to complex sentences as follows:[10]

$$v(\neg A) \ni \begin{cases} 1, & \text{if } 0 \in v(A) \\ 0, & \text{if } 1 \in v(A) \end{cases}$$

$$v(A \wedge B) \ni \begin{cases} 1, & \text{if } 1 \in v(A) \text{ and } 1 \in v(B) \\ 0, & \text{if } 0 \in v(A) \text{ or } 0 \in v(B) \end{cases}$$

$$v(A \vee B) \ni \begin{cases} 1, & \text{if } 1 \in v(A) \text{ or } 1 \in v(B) \\ 0, & \text{if } 0 \in v(A) \text{ and } 0 \in v(B) \end{cases}$$

$$v(A \supset B) \ni \begin{cases} 1, & \text{if } 0 \in v(A) \text{ or } 1 \in v(B) \\ 0, & \text{if } 1 \in v(A) \text{ and } 0 \in v(B) \end{cases}$$

Having defined all 4-valued valuations, we can now define the set of admissible valuations for each of the logics in the FDE family: CL, which allows neither gaps nor gluts; K3, which allows gaps but no gluts; LP, which allows gluts but no gaps; and FDE, which allows both gaps and gluts:

> **Admissible Valuations:** The admissible valuations for CL, LP, K3, and FDE are the subsets of the above set of valuations where atomics are mapped to only a certain subset of the four truth-values:
>
> 1. **CL:** All valuations $\mathcal{A} \to \{\{1\}, \{0\}\}$
> 2. **K3:** All valuations $\mathcal{A} \to \{\varnothing, \{1\}, \{0\}\}$
> 3. **LP:** All valuations $\mathcal{A} \to \{\{1\}, \{0\}, \{1, 0\}\}$
> 4. **FDE:** All valuations $\mathcal{A} \to \{\varnothing, \{1\}, \{0\}, \{1, 0\}\}$

Now, *unilateral validity* is preservation of *truth*, and the unilateral consequence relations of these logics are very well-studied. Our concern, however, will be with

[10] These are, of course, the familiar truth and falsity conditions from classical logic, just generalized to allow for the possibility of truth-value gaps and gluts. The general approach to four-valued semantics here is owed to Dunn [11]. For discussion, see [28, 74-76] and [4].

bilateral validity, which following Smiley and Rumfitt, we'll take to be preservation of *correctness*. Officially, we define it as follows:[11]

> **Bilateral Validity:** An argument of the form $\Gamma \vdash \varphi$ is *bilaterally valid*, relative to a set of admissible valuations V, $\Gamma \vDash_{B_V} \varphi$, just in case there is no $v \in V$ such that all of the stances in Γ are correct and φ is incorrect.

The four bilateral natural deduction systems, obtained just by modifying the coordination principles, are the following:

1. $\mathbf{BN_{CL}}$**:** Operational rules + Explosion + Excluded Middle
2. $\mathbf{BN_{K3}}$**:** Operational rules + Explosion
3. $\mathbf{BN_{LP}}$**:** Operational rules + Excluded Middle
4. $\mathbf{BN_{FDE}}$**:** Operational rules

It is easy to show that all systems are sound and complete with respect to bilateral validity. BN_{FDE} proves $\Gamma \vdash \varphi$ just in case $\Gamma \vDash_{\text{B}_{\text{FDE}}} \varphi$. Likewise for BN_{LP} and BN_{K3}

In order to systematically investigate the bilateral consequence relations of these bilateral systems for the FDE family, it will prove useful to generalize the notion of bilateral validity in a natural way to *multiple conclusion* arguments as follows:

> **Bilateral Validity (Multiple Conclusion):** An argument of the form $\Gamma \vdash \Delta$ is *bilaterally valid*, relative to a set of admissible valuations V, $\Gamma \vDash_{B_V} \Delta$, just in case there is no $v \in V$ such that all of the stances in Γ are correct and all of the stances in Δ are incorrect.

So, in addition to these natural deduction systems, let us introduce a family of bilateral multiple conclusion sequent systems:[12]

$$\frac{}{\Gamma, \varphi \vdash \varphi, \Delta}\ \text{Reflex.} \qquad \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ \text{Weak.} \qquad \frac{\Gamma, \varphi \vdash \Delta \quad \Gamma' \vdash \varphi, \Delta'}{\Gamma, \Gamma' \vdash \Delta', \Delta}\ \text{Cut}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta}{\Gamma, +\langle \neg A\rangle \vdash \Delta}\ +_{\neg_L} \qquad \frac{\Gamma, +\langle A\rangle \vdash \Delta}{\Gamma, -\langle \neg A\rangle \vdash \Delta}\ -_{\neg_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta}{\Gamma \vdash +\langle \neg A\rangle, \Delta}\ +_{\neg_R} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta}{\Gamma \vdash -\langle \neg A\rangle, \Delta}\ -_{\neg_R}$$

[11] This is simply a generalization of what Rumfitt [36, 224-225] [37, 808] calls "Smiley consequence" beyond just classical valuations.

[12] These sequent systems are technically quite close to the 4-signed/4-sided systems that have been put forward by Wintein [49] and Shapiro [39], but, I would argue, conceptually superior in being straightforwardly intuitively intelligible in terms of the conception of bilateral validity just laid out.

$$\frac{\Gamma, +\langle A\rangle, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \wedge B\rangle \vdash \Delta}\ +_{\wedge_L} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta \quad \Gamma \vdash +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \wedge B\rangle, \Delta}\ +_{\wedge_R}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta \quad \Gamma, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \wedge B\rangle \vdash \Delta}\ -_{\wedge_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \wedge B\rangle, \Delta}\ -_{\wedge_R}$$

$$\frac{\Gamma, +\langle A\rangle \vdash \Delta \quad \Gamma, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \vee B\rangle \vdash \Delta}\ +_{\vee_L} \qquad \frac{\Gamma \vdash +\langle A\rangle, +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \vee B\rangle, \Delta}\ +_{\vee_R}$$

$$\frac{\Gamma, -\langle A\rangle, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \vee B\rangle \vdash \Delta}\ -_{\vee_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, \Delta \quad \Gamma \vdash -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \vee B\rangle, \Delta}\ -_{\vee_R}$$

$$\frac{\Gamma, -\langle A\rangle \vdash \Delta \quad \Gamma, +\langle B\rangle \vdash \Delta}{\Gamma, +\langle A \supset B\rangle \vdash \Delta}\ +_{\supset_L} \qquad \frac{\Gamma \vdash -\langle A\rangle, +\langle B\rangle, \Delta}{\Gamma \vdash +\langle A \supset B\rangle, \Delta}\ +_{\supset_R}$$

$$\frac{\Gamma, +\langle A\rangle, -\langle B\rangle \vdash \Delta}{\Gamma, -\langle A \supset B\rangle \vdash \Delta}\ -_{\supset_L} \qquad \frac{\Gamma \vdash +\langle A\rangle, \Delta \quad \Gamma \vdash -\langle B\rangle, \Delta}{\Gamma \vdash -\langle A \supset B\rangle, \Delta}\ -_{\supset_R}$$

We will now give the coordination principles as the following axiom schemas that one might have in addition to Reflexivity

$$\frac{}{\Gamma, \varphi, \varphi^* \vdash \Delta}\ \text{Explo.} \qquad \frac{}{\Gamma \vdash \varphi, \varphi^*, \Delta}\ \text{Ex. Mid.}$$

Considering all of the possibilities for coordination principles, we'll define the following four sequent systems:

1. $\mathbf{BS_{CL}}$**:** Operational rules + Explosion and Excluded Middle
2. $\mathbf{BS_{LP}}$**:** Operational rules + Excluded Middle
3. $\mathbf{BS_{K3}}$**:** Operational rules + Explosion
4. $\mathbf{BS_{FDE}}$**:** Operational rules

Just a few words on these sequent systems: Like Ketonen's [21] classical sequent calculus, proof of a sequent is constructed by root-first proof search, and this yields a decision procedure for proving or refuting sequents. Notably, both Cut and Weakening are eliminable, and Reflexivity (as well as Excluded Middle and Explosion in the systems that have them) can be restricted to atomics. These proof-theoretic results, as well as soundness and completeness, are provided in the appendix.

Having introduced these bilateral systems, let us turn our attention to the coordination principles, which are the most notable feature of bilateral systems. I said above that coordination principles were "bilateral structural rules." At this point, it is worth being explicit about exactly what is meant by that. They are structural rules in that, unlike the operational rules, they don't involve any specific logical vocabulary. However, they are distinct from familiar structural rules like Weakening and Cut in that those structural rules impose structure on *the consequence relation*. Rather than imposing structure on the consequence relation, coordination principles impose structure on *the relation between affirmation and denial*. Since we already have the term "coordination principles" for structural rules of the former sort, let us reserve the term "structural rules" specifically for familiar structural rules of the latter sort. In a bilateral context, understanding consequence as correctness-preservation, all bilateral systems are "fully structural" in that the consequence relations of all of these logics are fully structural.

Given structural rules, many coordination principles are inter-derivable. For instance, in the ND system, the coordination principles of Explosion and Excluded Middle were presented at the *meta-inferential* level, relating sequents, here they are presented at the (first-order) *inferential* level. In this multiple conclusion context, it is worth considering the multiple conclusion generalizations of metainferential Explosion and Excluded Middle:

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi^*}{\Gamma \vdash \Delta}\ \text{Explo.} \qquad\qquad \frac{\Gamma, \varphi \vdash \Delta \quad \Gamma, \varphi^* \vdash \Delta}{\Gamma \vdash \Delta}\ \text{Ex. Mid}$$

Note that given Cut, the first-order inferential principles are derivable from the meta-inferential principles and vice versa:

$$\dfrac{\dfrac{}{\Gamma, \varphi, \varphi^* \vdash \varphi}\ \text{Reflex.} \quad \dfrac{}{\Gamma, \varphi, \varphi^* \vdash \varphi^*}\ \text{Reflex.}}{\Gamma, \varphi, \varphi^* \vdash \Delta}\ \text{Explo.} \qquad \dfrac{\dfrac{}{\Gamma, \varphi \vdash \varphi, \varphi^*, \Delta}\ \text{Reflex.} \quad \dfrac{}{\Gamma, \varphi^* \vdash \varphi, \varphi^*, \Delta}\ \text{Reflex.}}{\Gamma \vdash \varphi, \varphi^*, \Delta}\ \text{Ex. Mid.}$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \varphi, \varphi^* \vdash \Delta}\ \text{Explo.} \quad \Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash \Delta}\ \text{Cut} \quad \Gamma \vdash \varphi^*}{\Gamma \vdash \Delta}\ \text{Cut} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \vdash \varphi, \varphi^*, \Delta}\ \text{Ex. Mid.} \quad \Gamma, \varphi^* \vdash \Delta}{\Gamma \vdash \varphi, \Delta}\ \text{Cut} \quad \Gamma, \varphi \vdash \Delta}{\Gamma \vdash \Delta}\ \text{Cut}$$

So, we could equally formulate these systems as involving the meta-inferential coordination principles rather than the first-order coordination principles. Another pair of coordination principles worth considering, which are likewise capable of playing the same role in the context of structural rules, are two that I call "In" and "Out" respectively:

$$\frac{\Gamma \vdash \varphi, \Delta}{\Gamma, \varphi^* \vdash \Delta}\text{ In} \qquad\qquad \frac{\Gamma, \varphi \vdash \Delta}{\Gamma \vdash \varphi^*, \Delta}\text{ Out}$$

It is easy to see that Excluded Middle is immediately derivable from Reflexivity with Out and Explosion is immediately derivable from Reflexivity with In. Conversely, given Cut, Out is immediately derivable from Excluded Middle and In is immediately derivable from Explosion:

$$\frac{\Gamma, \varphi \vdash \Delta \quad \dfrac{}{\Gamma \vdash \varphi, \varphi^*, \Delta}\text{ Ex. Mid.}}{\Gamma \vdash \varphi^*, \Delta}\text{ Cut} \qquad\qquad \frac{\Gamma \vdash \varphi, \Delta \quad \dfrac{}{\Gamma, \varphi, \varphi^* \vdash \Delta}\text{ Explo.}}{\Gamma, \varphi^* \vdash \Delta}\text{ Cut}$$

Thus, rather than adding Excluded Middle or Explosion as an additional axiom schema, another way to get a logic with the consequence relation of Bilateral LP or Bilateral K3 is with the addition of coordination principles is to add Out or In respectively.[13] However, there are proof-theoretic benefits of our treatment of coordination principles as axiom schemas, as it facilitates root-first proof search, making the completeness proofs straightforward.

In and Out together yield (the multiple conclusion version of) the principle that Smiley [42] dubs Reversal:

$$\frac{\Gamma, \varphi \vdash \psi, \Delta}{\Gamma, \psi^* \vdash \varphi^*, \Delta}\text{ Reversal}$$

Where $\{\varphi\}$ or $\{\psi\}$ can be empty.

It's obvious that, with this rule, having both left rules and right rules is redundant, since one can put forward calculus containing only rules for deriving formulas on one side of the turnstile and be able to derive any formula on the other side the turnstile by deriving its opposite and using Reversal. Thus, given that Reversal is derivable in BS_{CL}, half of the rules are redundant. In the bilateral sequent calculus for classical logic proposed by Simonelli [40], Reversal is the only coordination principle, and there are only right rules: one positive rule and one negative rule for each connective. For the non-classical members of the FDE family, however, there is a crucial asymmetry between the two sides of the turnstile, and so having both left and right rules is necessary.

[13] This is the route that Shapiro [39] takes in his proposal of a 4-sided sequent calculus for the FDE family where In and Out correspond to what he calls "shift" rules.

4 Reasoning with the Liar

We are now in a position to return to formalize the informal reasoning about the liar presented in Section 2. Let us supplement our bilateral systems with the following rules for the truth predicate *Tr* such that affirming $Tr\ulcorner A\urcorner$ is correct just in case affirming A is correct and denying $Tr\ulcorner A\urcorner$ is correct just in case denying A is correct:[14]

$$\frac{\Gamma \vdash +\langle A\rangle}{\Gamma \vdash +\langle Tr\ulcorner A\urcorner\rangle}\ {+}Tr_R \qquad\qquad \frac{\Gamma \vdash -\langle A\rangle}{\Gamma \vdash -\langle Tr\ulcorner A\urcorner\rangle}\ {-}Tr_R$$

Let us further supplement our systems with rules for affirming and denying the sentence ℓ, which says of itself that it's not true:

$$\frac{\Gamma \vdash +\langle \neg Tr\ulcorner \ell\urcorner\rangle}{\Gamma \vdash +\langle \ell\rangle}\ {+}\ell_R \qquad\qquad \frac{\Gamma \vdash -\langle \neg Tr\ulcorner \ell\urcorner\rangle}{\Gamma \vdash -\langle \ell\rangle}\ {-}\ell_R$$

With these rules, we can reason as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{+\langle \ell\rangle \vdash +\langle \ell\rangle}{+\langle \ell\rangle \vdash +\langle Tr\ulcorner \ell\rangle}\ {+}Tr_R}{+\langle \ell\rangle \vdash -\langle \neg Tr\ulcorner \ell\urcorner\rangle}\ {-}_{\neg R}}{+\langle \ell\rangle \vdash -\langle \ell\rangle}\ {-}\ell_R}{\vdots} \qquad\qquad \dfrac{\dfrac{\dfrac{\dfrac{-\langle \ell\rangle \vdash -\langle \ell\rangle}{-\ell \vdash -\langle Tr\ulcorner \ell\rangle\urcorner}\ {-}Tr_R}{-\langle \ell\rangle \vdash +\langle \neg Tr\ulcorner \ell\urcorner\rangle}\ {+}_{\neg R}}{-\langle \ell\rangle \vdash +\langle \ell\rangle}\ {+}\ell_R}{\vdots}$$

The proof on the left reads as follows. Affirming the liar commits one to affirming the liar. So, affirming the liar commits one to affirming that the liar is true. Accordingly, affirming the liar commits one to denying that the liar is not true. But "that the liar is not true" is just what the liar says. So affirming the liar commits one to denying the liar. The proof on the right reads as follows. Denying the liar commits one to denying the liar, and so denying the liar commits one to denying that the liar is true. Accordingly, denying the liar commits one to affirming that the liar is not true. But "that the liar is not true" is just what the liar says, and so denying the liar commits one to affirming the liar. All of this reasoning seems impeccable. Moreover, it seems like the sequents we end up with here express exactly what we want to say about the liar; the liar is a sentence such that affirming it commits one

[14] I introduce just (invertible) right rules for the moment, so as to remain neutral between natural deduction and sequent setting. The corresponding left rules appealed to in the next section are obvious.

to denying it and denying it commits one to affirming it. There may be things one wants to say beyond that, but that much, it seems, is undeniable.[15]

The logical steps involved in the above reasoning—the instances of Reflexivity and the uses of the negation rules—are valid in all four bilateral systems in the FDE family.[16] However, given that more inferences can be made in B_{K3} and B_{LP} than can be made in B_{FDE} in virtue of the presence of coordination principles, we can consider what further reasoning with the liar we can do in these two logics. In particular, let us consider again the principles of Out and In but where Δ is empty:

$$\dfrac{\Gamma, \varphi \vdash}{\Gamma \vdash \varphi^*}\ \text{Out} \qquad\qquad \dfrac{\Gamma \vdash \varphi}{\Gamma, \varphi^* \vdash}\ \text{In}$$

This instance of Out says that if Γ along with some stance φ is *incoherent*, then Γ *commits one* to the opposite stance, φ^*, whereas this instance of In says that if Γ commits one to some stance φ, then Γ along with φ^* is incoherent. Recall that Out is valid in B_{LP} but invalid in B_{K3} whereas In is valid in B_{K3} but invalid in B_{LP}. So, B_{K3} lets us reason from the conclusion of the above two proofs as follows:[17]

$$\dfrac{\dfrac{+\langle \ell \rangle \vdash -\langle \ell \rangle}{+\langle \ell \rangle, +\langle \ell \rangle \vdash}\ \text{In}}{+\langle \ell \rangle \vdash}\ \text{Cont.} \qquad\qquad \dfrac{\dfrac{-\langle \ell \rangle \vdash +\langle \ell \rangle}{-\langle \ell \rangle, -\langle \ell \rangle \vdash}\ \text{In}}{-\langle \ell \rangle \vdash}\ \text{Cont.}$$

We read these above two proofs as follows. Affirming the liar commits one to denying the liar. Accordingly, it's incoherent to affirm the liar. Likewise, denying the liar commits one to affirming the liar. Accordingly, it's incoherent to deny the liar. Thus, it's incoherent to affirm the liar, and it's incoherent to deny the liar. B_{LP} on the other hand, lets us reason as follows:

[15] It's important to emphasize that the consequence relation at play here is one of *commitment* preservation, rather than *entitlement* preservation. See Incurvati and J. Schlöder [18] on this point, who argue that the truth-rules here preserve commitment but not evidence.

[16] For the non-logical steps to be valid, we suppose that the semantics of our truth-predicate Tr is such that $v(Tr\ulcorner A \urcorner)$ always equals $v(A)$, and we suppose that $v(\ell)$ always equals $v(\neg Tr\ulcorner \ell \urcorner)$. Given this, it is easy to see that $v(\ell)$ must be either $\varnothing$ or $\{1, 0\}$ in all FDE valuations. If $1 \in v(\ell)$, then $1 \in v(Tr\ulcorner \ell \urcorner)$, and so $0 \in v(\neg Tr\ulcorner \ell \urcorner)$, and so $0 \in v(\ell)$. Likewise if $0 \in v(\ell)$, then $0 \in v(Tr\ulcorner \ell \urcorner)$, and so $1 \in v(\neg Tr\ulcorner \ell \urcorner)$, and so $1 \in v(\ell)$. So, $v(\ell)$ must be either $\varnothing$ or $\{1, 0\}$ in all FDE valuations. This means that it must be $\varnothing$ in all K3 valuations, it must be $\{1, 0\}$ in all LP valuations, and there is no value it can possibly take in any CL valuation.

[17] I show Contraction for clarity, but, technically, it is built into out treatment of what flanks the two sides of the turnstile as sets

$$\dfrac{\dfrac{+\langle\ell\rangle \vdash -\langle\ell\rangle}{\vdash -\langle\ell\rangle, -\langle\ell\rangle}\text{ Out}}{\vdash -\langle\ell\rangle}\text{ Cont.} \qquad \dfrac{\dfrac{-\langle\ell\rangle \vdash +\langle\ell\rangle}{\vdash +\langle\ell\rangle, +\langle\ell\rangle}\text{ Out}}{\vdash +\langle\ell\rangle}\text{ Cont.}$$

According to B_{LP}, affirming the liar commits one to denying the liar, and denying the liar commits one to affirming the liar, and so one is committed both to affirming the liar and to denying the liar. So, whereas, in the case of B_{K3}, both of the two opposite stances towards the liar are incorrect, in the case of B_{LP}, both stances are correct.

Consider now the instances of (metainferential) Explosion and Excluded Middle where Δ is empty:

$$\dfrac{\Gamma \vdash \varphi \quad \Gamma \vdash \varphi^*}{\Gamma \vdash}\text{ Explo.} \qquad \dfrac{\Gamma, \varphi \vdash \quad \Gamma, \varphi^* \vdash}{\Gamma \vdash}\text{ Ex. Mid}$$

This instance of Explosion (which might be dubbed *Bilateral Non-Contradiction* (c.f. [37, 804], [18, 67])), says that if a set of stances Γ commits one to some stance φ and Γ also commits one to the opposite stance φ, then Γ is incoherent. The instance of Excluded Middle (which, for reasons I'll make clear below, can be understood as a principle of *Extensibility* (c.f. [30, 9] [33, 143])) says that if Γ along with some stance φ is incoherent and Γ along with the opposite stance φ^* is incoherent, then Γ must itself be incoherent. Once again, B_{LP} rejects Explosion, and good thing that it does, since, if Explosion were valid in B_{LP}, it could be applied at the end of the above proof to yield the empty sequent (and so every sequent, via Weakening). Likewise, B_{K3} rejects Excluded Middle, and, once again, good thing that it does, since Excluded Middle could be applied at the end of the above proof to yield the empty sequent (and so every sequent via Weakening).

These two non-classical systems make formally precise the two responses to the liar articulated in Section 2. A proponent of B_{K3} maintains the liar is such that it is never correct to either affirm it or deny it, whereas a proponent of B_{LP} maintains that the liar is such that it is always correct to both affirm it and deny it. Moreover, a proponent of B_{FDE} may acknowledge both of these possibilities but wish to stay neutral as to which is correct.

5 (Bilaterally Interpreted) ST is (Solely Left-Sided) B_{K3}

I have articulated bilateral systems that make explicit the *neither* and *both* responses to the liar. Now, I've presented these systems as *subclassical* systems, since the positively signed fragments of the consequence relations of these systems align with the unilateral consequence relations of familiar subclassical logics. However, it's worth emphasizing at this point that the positively signed fragments of these bilateral consequence relations are just that: *fragments*. For instance, not only does the Bilateral K3 consequence relation contain $+\langle p \vee q\rangle, +\langle\neg p\rangle \vDash +\langle q\rangle$ and $+\langle p \vee q\rangle, +\langle\neg p\rangle, +\langle\neg q\rangle \vDash$, but it also contains $+\langle p \vee q\rangle, -\langle p\rangle \vDash +\langle q\rangle$ and $+\langle p \vee q\rangle, -\langle p\rangle, -\langle q\rangle \vDash$. This raises the question of how, exactly, to relate these bilateral classical consequence relations to more familiar unilateral consequence relations. In the remainder of the paper, I will focus in particular on Bilateral K3, as this logic, I believe, has occupied a central place in recent debates surrounding the liar, though it has not been explicitly formulated in these terms. In order to appreciate the significance of this bilateral system in the context of such contemporary debates, let us turn for a moment to consider a pair of *uni*lateral consequence relations before turning back to the *bi*lateral consequence relation of K3.

Given the set of K3 valuations, there are two notions of validity one might naturally define. The first is the standard notion of validity as truth-preservation. In general, an argument of the form $X \vdash Y$ is valid, relative to a set of valuations V, $X \vDash_V Y$, just in case there's no valuation in V such that all of the sentences in X are true and all of the sentences in Y are untrue. So, for K3, this is the following:

> **K3 Validity:** $X \vDash_{K3} Y$ just in case there is no $v \in$ K3 such that $1 \in v(A)$ for all $A \in X$ and $1 \notin v(B)$ for all $B \in Y$.

However, recent work has shown that one can take the same set of K3 valuations and define a different notion of validity—known as *ST*-Validity—according to which an argument of the form $X \vdash Y$ is valid just in case there's no valuation such that all of the sentences in X are true and all of the sentences in Y are false:[18]

> **ST-Validity:** $X \vDash_{ST} Y$ just in case there is no $v \in K3$ such that $1 \in v(A)$ for all $A \in X$ and $0 \in v(B)$ for all $B \in Y$.

The interesting fact about ST-consequence is that it aligns completely with the familiar consequence relation of classical logic in the sense that $X \vDash_{ST} Y$ just in case

[18] It's worth being clear that one can define the same notion of validity with the set of LP valuations (note that the three-valued truth tables with either $\varnothing$ or $\{1, 0\}$ are the same), and many presentations of ST use $\frac{1}{2}$ as a middle value that can be interpreted either in a gappy way or a glutty way. I define ST in terms of K3 validity because of the philosophical interpretation to follow. It's perhaps also worth noting that, following standard practice, I'm referring to this specific three-valued logic as "ST" here, but this is not the only Strict/Tolerant Logic. Fitting [14] [13] shows that there are number of strict/tolerant which align with familiar logics (e.g. LP or K3) at the first-order inferential level but depart at the meta-inferential level in the same way.

$X \vDash_{CL} Y$. However, certain *meta*-inferences that are valid in CL are invalid in ST. Most notably, consider the rule of *Cut*, the basic *Transitivity* rule for a consequence relation:[19]

$$\frac{X, A \vdash Y \quad X \vdash A, Y}{X \vdash Y}\,\text{Cut}$$

Suppose $v(A)$ is $\varnothing$ on all valuations. Then the two premises are valid since there's no valuation in A is true nor is there any valuation in which A is false, but there may well be a valuation in which everything in X is true and everything in Y is false. This opens the door for a "substructural" solution to the liar paradox, which accepts transparent truth, accepts all of the operational rules of the standard classical sequent calculus, but rejects the structural rule Cut. In particular, one can accepting the standard negation rules:

$$\frac{X \vdash A, Y}{X, \neg A \vdash Y}\,\text{L}_{\neg} \qquad\qquad \frac{X, A \vdash Y}{X \vdash \neg A, Y}\,\text{R}_{\neg}$$

but, rejecting Cut, one blocks the following triviality proof at the final step:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X, \ell \vdash \ell, Y}\,\text{Reflex.}}{X, \ell \vdash Tr\ulcorner \ell \urcorner, Y}\,\text{Tr}_R}{X, \ell, \neg Tr\ulcorner \ell \urcorner \vdash Y}\,\neg_L}{X, \ell, \ell \vdash Y}\,\ell_L}{X, \ell \vdash Y}\,\text{Contr}_L \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{X, \ell \vdash \ell, Y}\,\text{Reflex.}}{X, Tr\ulcorner \ell \urcorner \vdash \ell, Y}\,\text{Tr}_L}{X \vdash \neg Tr\ulcorner \ell \urcorner, \ell, Y}\,\neg_R}{X \vdash \ell, \ell, Y}\,\ell_R}{X \vdash \ell, Y}\,\text{Contr}_R}{X \vdash Y}\,\text{Cut (invalid step)}$$

It is no exaggeration to say that this approach to the liar is the most significant formal development concerning semantic paradoxes in decades. Since the origin of technical work on the paradoxes, it was thought that, if one wants a transparent truth predicate, one must give up some classical inferences. The ST-based solution promises to enable one to have one's cake and eat it too.

Of course, it is one thing for a solution to the liar paradox to be technically possible and another thing for it to be philosophically plausible. On the latter front, the most work to philosophically motivate the ST-based solution the liar has come by way of David Ripley. Ripley's [33] key idea is to motivate ST-validity by appealing to the bilateral interpretation of multiple conclusion sequents proposed by Restall [30]. On this reading, a multiple conclusion sequent of the form $X \vdash Y$

[19] For the present purposes, it does no harm to simply align Cut and Transitivity. However, see [35] for a more careful discussion of various Transitivity principles and their relations.

is understood as saying that the position consisting *affirming* everything in X and *denying* everything in Y is incoherent, or, as Ripley puts it, "out of bounds."[20] Deploying the sort of explicitly bilateral notation used here, we might express the position consisting affirming everything in X and denying everything in Y as $+\langle X\rangle, -\langle Y\rangle$ (where $+\langle X\rangle$ is shorthand for $\{+\langle A\rangle \mid A \in X\}$). So, for instance, on this interpretation of the multiple conclusion sequent calculus, the axioms schema $X, A \vdash A, Y$ says that relative to any position $+\langle X\rangle, -\langle Y\rangle$, affirming and denying any sentence A is out of bounds. The negation rules say that relative to any position $+\langle X\rangle, -\langle Y\rangle$, if affirming A is out of bounds, then denying $\neg A$ is out of bounds, and, likewise, if denying A is out of bounds, then affirming $\neg A$ is out of bounds. Finally, Cut, says that if, relative to some position $+\langle X\rangle, -\langle Y\rangle$, affirming A is out of bounds, and, relative to that position, denying A is also out of bounds, then that position must itself be out of bounds. Restall articulates Cut, so understood, as a principle of *Extensibility*: it says that, for any position $+\langle X\rangle, -\langle Y\rangle$ and any sentence A, if $+\langle X\rangle, -\langle Y\rangle$ is coherent, then it must be *coherently extensible* to either $+\langle X\rangle, -\langle Y\rangle, +\langle A\rangle$ or $+\langle X\rangle, -\langle Y\rangle, -\langle A\rangle$. Though Restall accepts this principle, Ripley argues that if A is a paradoxical sentence such as the liar, then, relative to any position, it is always incoherent to affirm A, but it's also incoherent to deny A. Cut/Extensibility would enable one to conclude from this fact that every position is incoherent, which, of course, cannot be accepted. Interpreting the above proof in this fashion, it becomes quite natural to take the final step to be problematic one. Moreover, given this conception of what a multiple conclusion sequent says, then the only thought needed to motivate the ST conception of validity is that affirming everything in X along with denying everything in Y is out of bounds just in case there's no valuation according to which everything in X is true and everything in Y is false.

In this way, Ripley manages to make the "non-transitive" solution to the liar paradox plausible. It comes, however, at the cost of articulating "logical consequence" in such a way that many would deny that it is really a relation of *consequence* at all. Standard conceptions of logical consequence, alethic or normative, understand consequence as a *preservation* relation, whether this is preservation of *truth*, as it is in standard developments of logic, or preservation of *commitment*, as it is in normative inferentialist developments of logic, and, as Beall, Glanzberg, and Ripley [5] point out, "at least on the usual way of understanding it, preservation is transitive," (104). Prevailing wisdom, then, is that, if one accepts Ripley's solution

[20]Generally, Restall and Ripley speak in terms of "assertion" and "denial" rather than "affirmation" and "denial," but nothing hangs on this difference.

to the liar, then one needs to reject such a conception of consequence. We are now in a position to see that this is not so, for Ripley's solution to the liar is just that of Bilateral K3, but restricted to the solely left-sided fragment of the consequence relation. Concretely, we can translate back and forth between multiple conclusion unilateral sequents, interpreted in bilateral fashion, and solely left-sided bilateral sequents of the sort that figure in the systems put forward here:

> **Translation Schema:** To translate a unilateral multiple conclusion sequent of the form $X \vdash Y$ to a bilateral sequent of the form $\Gamma \vdash$, let $\Gamma = \{+\langle A \rangle \mid A \in X\} \cup \{-\langle B \rangle \mid B \in Y\}$. Conversely, to translate a sequent of the form $\Gamma \vdash$ to a sequent of the form $X \vdash Y$, let $X = \{A \mid +\langle A \rangle \in \Gamma\}$ and $Y = \{B \mid -\langle B \rangle \in \Gamma\}$.

This makes Restall and Ripley's bilateral interpretation of multiple conclusion sequents *explicit in the formal system itself*. For Restall and Ripley, the role of the turnstile is simply to partition the set of assertions from the set of denials. In the bilateral framework put forward here, the positive and negative signing of formulas does this job (and does it much more perspicuously), and this frees up the turnstile to do the job it is supposed to do: codify *consequence*. In particular, by going explicitly bilateral and endorsing Bilateral K3, we are able to endorse a system that contains an account of both the *incoherence* of sets of affirmations and denials, codified by solely left-sided sequents, and relations of *committive consequence* between affirmations and denials, codified by sequents that have formulas on the righthand side, where the relation between these two notions is formally codified by the coordination principles accepted (and rejected) by Bilateral K3.

To spell out the above thought, consider first the solely left-sided fragment of BS_{K3}, containing the axiom schema of Bilateral Explosion where Δ is null and the set of left rules where Δ is null. Applying the translation schema, it is easy to see that this fragment is simply a notational variant of the unilateral sequent calculus for classical logic.[21] For instance, the translations of the standard unilateral negation rules shown above are the following:

$$\frac{\Gamma, -\langle A \rangle \vdash}{\Gamma, +\langle \neg A \rangle \vdash}\ +_{\neg_L} \qquad\qquad \frac{\Gamma, +\langle A \rangle \vdash}{\Gamma, -\langle \neg A \rangle \vdash}\ -_{\neg_L}$$

These translated negation rules now *show*, explicitly in the notation itself, what the familiar unilateral negation rules *say*, on Restall and Ripley's bilateral interpretation

[21] In particular, it is a translation of Ketonen's [21] version of the classical sequent calculus, in which the rules for conjunction and disjunction are symmetric.

of them. In general, the left rules of the unilateral sequent calculus correspond to the positive rules of the solely left-sided fragment of BSK3 and the right rules correspond to the negative rules. In this way, the sequent system for Bilateral K3 contains the standard sequent system for Unilateral CL as its solely left-sided fragment, with Restall and Ripley's bilateral interpretation built in. Now, of course, so does the sequent system for Bilateral CL. Crucially, however, though the solely left-sided fragments of these systems align at the level of *inferences*, they diverge at the level of *meta-inferences*. In particular, as we already noted in the previous section, Bilateral K3 rejects the following instance of Excluded Middle:

$$\frac{\Gamma, \varphi \vdash \quad \Gamma, \varphi^{*} \vdash}{\Gamma \vdash}\ \text{Ex. Mid}$$

Applying the translation schema once more, it is easy to see that this just is Unilateral Cut, once again with the bilateral interpretation made explicit. So, translating the above unilateral proof, which gets blocked at the final step, we have the following:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, +\langle \ell \rangle, -\langle \ell \rangle \vdash}\ \text{Explo.}}{\Gamma, +\langle \ell \rangle, -\langle Tr\ulcorner \ell \urcorner \rangle \vdash}\ -_{Tr_L}}{\Gamma, +\langle \ell \rangle, +\langle \neg Tr\ulcorner \ell \urcorner \rangle \vdash}\ +_{\neg_L}}{\dfrac{\Gamma, +\langle \ell \rangle, +\langle \ell \rangle \vdash}{\Gamma, +\langle \ell \rangle \vdash}\ \text{Contr.}}\ +_{\ell_L} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma, -\langle \ell \rangle, +\langle \ell \rangle \vdash}\ \text{Explo.}}{\Gamma, -\ell, +\langle Tr\ulcorner \ell \rangle \urcorner \vdash}\ +_{Tr_L}}{\Gamma, -\langle \ell \rangle, -\langle \neg Tr\ulcorner \ell \urcorner \rangle \vdash}\ -_{\neg_L}}{\dfrac{\Gamma, -\langle \ell \rangle, -\langle \ell \rangle \vdash}{\Gamma, -\langle \ell \rangle \vdash}\ \text{Contr.}}\ -_{\ell_L}}{\Gamma \vdash}\ \text{Ex Mid. (invalid step)}$$

Having translated Ripley's solution into this bilateral framework, it's clear that it is just the solution of B_{K3} outlined above, but restricted to the left side of the turnstile, and, indeed, it's easy to see that the notion of bilateral validity we've defined for the solely left-sided fragment of B_{K3} just is that of ST. So this is indeed a faithful translation of Ripley's approach. The crucial upshot of this translation, however, is that we can now *extend this approach to the other side of the turnstile*, formally articulating not only which sets of stances it's *incoherent* to adopt, but also which stances one is *committed* to adopting, given one's adoption of various other stances. For instance, a proponent of Ripley's solution can accept all of the FDE reasoning shown in the previous section that formally codifies how affirming the liar commits one to denying the liar and denying the liar commits one to affirming the liar. Indeed, one can maintain that this is precisely *why* one shouldn't take a stance, positive or negative, on the liar. If you take one, you've got to take the other, and then you've taken opposite stances towards a single sentence, and so your position is incoherent. Our explicitly bilateral system thus formally captures aspects of the informal reasoning—left out by Ripley's system—that makes Ripley's solution to the liar so intuitively plausible.

Beyond just adding expressive power, this explicitly bilateral recasting of Ripley's ST-based solution to the liar has important implications for the debate over "non-classical" versus "substructural" approaches to paradox.[22] Ripley [33] claims that "the core of the advantages of the ST approach over K3T- and LPT-based approaches" is that it enables one to have a transparent truth-predicate while retaining all of classical logic (156). In a sense, this is true. All of *unilateral* classical logic is retained in this solution. In a deeper sense, however, the solution is not classical, since all of *bilateral* classical logic is not retained. For instance, even though BS_{K3} proves $-\langle A \vee \neg A \rangle \vdash$ (which, in Ripley's imperspicuous unilateral notation, would be written as $\vdash A \vee \neg A$), BS_{K3} does not prove $\vdash +\langle A \vee \neg A \rangle$. It makes sense, on this interpretation, why it should not: one should not affirm $\ell \vee \neg \ell$ if one wants to avoid affirming a contradiction, since affirming $\ell \vee \neg \ell$ commits one to affirming $\ell \wedge \neg \ell$.[23] The only way to extend Ripley's solution to the liar to the right side of the turnstile is by rejecting bilateral classicality. So, while Ripley says "there is no need, from an ST-based perspective, ever to criticize (on logical grounds) any classically-valid inference" (156), this is false insofar as we're considering inferences that are *bilaterally* classically valid. The same remarks apply for Ripley's advertisement of his approach as enabling one to have a workable material conditional. While it's true that BS_{K3} proves $-\langle A \supset A \rangle \vdash$, it's not the case that $\vdash +\langle A \supset A \rangle$.

On the flip side of things, Ripley takes his approach to be *substructural*. However, the consequence relation of Bilateral K3, along with all of the other logics presented here, is *fully structural*. Of course, as noted above, coordination principles are a kind of structural rule, and so, in that non-standard sense, the approach is "substructural." However, as far as the structure of the *consequence relation* is concerned, all the usual structural rules hold. Most importantly, the consequence relation of Bialteral K3 is completely transitive. One cannot derive both $+\langle A \rangle \vdash$ and $\vdash +\langle A \rangle$, and so there is no reason to restrict Cut. Thus, this recasting of the "non-

[22]For discussion of this debate, see [33], [34], and [38].

[23]The proof goes as follows:

$$\dfrac{\dfrac{\dfrac{}{+\langle \ell \rangle \vdash +\langle \ell \rangle}\,\text{Reflex.} \qquad \dfrac{\dfrac{\vdots}{-\langle \ell \rangle \vdash +\langle \ell \rangle}}{+\langle \neg\ell \rangle \vdash +\langle \ell \rangle}\,+_{\neg_L}}{+\langle \ell \vee \neg\ell \rangle \vdash +\langle \ell \rangle}\,+_{\vee_L} \qquad \dfrac{\dfrac{\dfrac{\vdots}{+\langle \ell \rangle \vdash -\langle \ell \rangle}}{+\langle \ell \rangle \vdash +\langle \neg\ell \rangle}\,+_{\neg_R} \qquad \dfrac{\dfrac{\dfrac{}{-\langle \ell \rangle \vdash -\langle \ell \rangle}\,\text{Reflex.}}{+\langle \neg\ell \rangle \vdash -\langle \ell \rangle}\,+_{\neg_L}}{+\langle \neg\ell \rangle \vdash +\langle \neg\ell \rangle}\,+_{\neg_R}}{+\langle \ell \vee \neg\ell \rangle \vdash +\langle \neg\ell \rangle}\,+_{\vee_L}}{+\langle \ell \vee \neg\ell \rangle \vdash +\langle \ell \wedge \neg\ell \rangle}\,+_{\wedge_R}$$

transitive" approach to paradox in this explicitly bilateral context enables one to restore transitivity to the turnstile, maintaining a conception of logical consequence as a relation of *preservation*. The twist here is that the sort of "preservation" at issue here is not preservation of *truth*, but preservation of *correctness*. Now, there might be other good reasons to reject transitivity in the context of a bilateral consequence relation.[24] However, from the perspective developed here, the liar paradox is not one of them.

6 Appendix: Technical Results

6.1 Generalized Formulation of Proof Systems

To simply the presentation of systems and proofs, I will formulate these bilateral systems using the generalized notation proposed by Simonelli [40] [41].[25] Rather than using + or − to state the rules, I'll use variables such as $\boldsymbol{a}$ and $\boldsymbol{b}$ to indicate signs that may be either + or − along with a function * that maps + to − and − to +. So, for any signed formula of the form $\boldsymbol{a}\langle\varphi\rangle$, where $\boldsymbol{a} \in \{+,-\}$, if $\boldsymbol{a} = +$ then $\boldsymbol{a}^* = -$, and if $\boldsymbol{a} = -$ then $\boldsymbol{a}^* = +$. This enables the binary connective rules for the proof systems to be stated in terms of a *schema* such that, for any assignment of signs to $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$, the rules for some binary connective are given. The binary connective rules of the natural deduction system are all instances of the following rule schema:

$$\frac{\boldsymbol{a}\langle A\rangle \quad \boldsymbol{b}\langle B\rangle}{\boldsymbol{c}\langle A\circ B\rangle}\ \boldsymbol{c}_{\circ\mathrm{I}} \qquad \frac{\boldsymbol{c}\langle A\circ B\rangle}{\boldsymbol{a}\langle A\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_1} \qquad \frac{\boldsymbol{c}\langle A\circ B\rangle}{\boldsymbol{b}\langle B\rangle}\ \boldsymbol{c}_{\circ\mathrm{E}_2}$$

$$\frac{\boldsymbol{a}^*\langle A\rangle}{\boldsymbol{c}^*\langle A\circ B\rangle}\ \boldsymbol{c}^*{}_{\circ\mathrm{I}_1} \qquad \frac{\boldsymbol{b}^*\langle B\rangle}{\boldsymbol{c}^*\langle A\circ B\rangle}\ \boldsymbol{c}^*{}_{\circ\mathrm{I}_2} \qquad \frac{\boldsymbol{c}^*\langle A\circ B\rangle \quad \begin{matrix}\overline{\boldsymbol{a}^*\langle A\rangle}^{\,u}\\ \vdots \\ \overline{\varphi}\end{matrix} \quad \begin{matrix}\overline{\boldsymbol{b}^*\langle B\rangle}^{\,v}\\ \vdots \\ \overline{\varphi}\end{matrix}}{\varphi}\ \boldsymbol{c}^*{}_{\circ\mathrm{E}}{}^{u,v}$$

I'll regard any connective introduced by way of rules of this form as a *primitive binary connective*. In total, the rules for the following connectives are given by the following assignments (where | is the Sheffer Stroke, ↓ is Peirce's arrow, and $\not\subset$ is the dual of the conditional):

[24] For instance, I have said nothing about the Curry paradox in the context of detachable non-material conditionals, and I leave open the possibility that in that context, structural rules such as Transitivity or Contraction might plausibly be called into question. See also Hlobil [16], Brandom [6], and Hlobil and Brandom [17] for reasons to reject transitivity in the context of modeling defeasible material inferences.

[25] This notation is similar to Smullyan's [44, 21-23] "unifying notation," but both more flexible and more conceptually transparent.

$\wedge$: $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = +$

$|$: $\boldsymbol{a} = +, \boldsymbol{b} = +, \boldsymbol{c} = -$

$\supset$: $\boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = -$

$\not\supset$: $\boldsymbol{a} = +, \boldsymbol{b} = -, \boldsymbol{c} = +$

$\vee$: $\boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = -$

$\downarrow$: $\boldsymbol{a} = -, \boldsymbol{b} = -, \boldsymbol{c} = +$

$\not\subset$: $\boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = +$

$\subset$: $\boldsymbol{a} = -, \boldsymbol{b} = +, \boldsymbol{c} = -$

The binary connective rules for the sequent systems are the following:

$$\frac{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}\langle A \circ B\rangle \vdash \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{L}}} \qquad \frac{\Gamma \vdash \boldsymbol{a}\langle A\rangle, \Delta \quad \Gamma \vdash \boldsymbol{b}\langle B\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}\langle A \circ B\rangle, \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{R}}}$$

$$\frac{\Gamma, \boldsymbol{a}^*\langle A\rangle \vdash \Delta \quad \Gamma, \boldsymbol{b}^*\langle B\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}^*\langle A \circ B\rangle \vdash \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{L}}} \qquad \frac{\Gamma \vdash \boldsymbol{a}^*\langle A\rangle, \boldsymbol{b}^*\langle B\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}^*\langle A \circ B\rangle, \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{R}}}$$

For the semantics, let us now define correctness conditions with the use of a function, as follows:

> **Correctness Function:** The *correctness function* [] is a function from $\{+, -\}$ to $\{1, 0\}$ mapping + to 1 and − to 0.

Officially, we can now define correctness as follows:[26]

> **Correctness:** Taking some stance $\boldsymbol{a}$ towards some sentence A, $\boldsymbol{a}\langle A\rangle$, is *correct*, relative to some valuation v, just in case $[\boldsymbol{a}] \in v(A)$.

Having defined this notation, the semantic clauses for all of the binary connectives can be stated in a single schematic clause as follows:

$$v(A \circ B) \ni \begin{cases} [\boldsymbol{c}], & \text{if } [\boldsymbol{a}] \in v(A) \text{ and } [\boldsymbol{b}] \in v(B) \\ [\boldsymbol{c}^*], & \text{if } [\boldsymbol{a}^*] \in v(A) \text{ or } [\boldsymbol{b}^*] \in v(B) \end{cases}$$

Thus, the following results apply not just to the proof systems given in the body of the paper, containing negation, conjunction, disjunction, and the material conditional, but also, for instance, the single connective truth-functionally complete systems containing just the Sheffer Stroke or just Peirce's Arrow.

6.2 Proof-Theoretic Properties

I will focus on the BS systems. For generalized harmony results regarding the rules of the BN systems (proposed as "BNK1" for classical logic), see [41].

[26] One can read the expression $[\boldsymbol{a}]$ as "the truth value that would make stance $\boldsymbol{a}$ correct."

Proposition 1: All axiom schemas of all BS systems can be restricted to atomics.

Proof: We consider Reflexivity first. The proof involves two inductions. For the first induction, we show that any sequent of the $\Gamma, \varphi \vdash \varphi, \Delta$, where φ is atomic, is derivable from solely atomic axioms by induction on complexity of the most complex formulas in Γ, Δ. This is trivial, as we always retain φ on both sides through any application of the connective rules. For the second induction, we show that any sequent of the form $\Gamma, \varphi \vdash \varphi, \Delta$ is derivable by induction on the complexity of φ. The base case is established by the first induction. For the inductive step, we suppose that φ is complexity $n + 1$ and show we can derive $\Gamma, \varphi \vdash \varphi$ from some number of sequents of the form $\Gamma', \psi \vdash \psi$ where ψ is complexity n. Where φ is of the form $\boldsymbol{c}\langle A \circ B\rangle$ the following derivation establishes this:

$$\dfrac{\dfrac{\Gamma, \boldsymbol{a}, \langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \boldsymbol{a}\langle A\rangle, \Delta \quad \Gamma, \boldsymbol{a}, \langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \boldsymbol{b}\langle B\rangle, \Delta}{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \boldsymbol{c}\langle A \circ B\rangle, \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{R}}}}{\Gamma, \boldsymbol{c}\langle A \circ B\rangle \vdash \boldsymbol{c}\langle A \circ B\rangle, \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{L}}}$$

The case where φ is of the form $\boldsymbol{c}^*\langle A \circ B\rangle$ is exactly dual. Now, consider Explosion. The first induction is exactly the same. For the inductive step of the second induction, we show that we can derive $\Gamma, \varphi, \varphi^* \vdash \Delta$ from some number of sequents of the form $\Gamma', \psi, \psi^* \vdash \Delta$. There is just one case to consider:

$$\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle, \boldsymbol{a}^*\langle A\rangle \vdash \Delta \quad \Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle, \boldsymbol{b}^*\langle B\rangle \vdash \Delta}{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle, \boldsymbol{c}^*\langle A \circ B\rangle \vdash \Delta}\ \boldsymbol{c}^*{}_{\circ_{\mathrm{L}}}}{\Gamma, \boldsymbol{c}\langle A \circ B\rangle, \boldsymbol{c}^*\langle A \circ B\rangle \vdash \Delta}\ \boldsymbol{c}_{\circ_{\mathrm{L}}}$$

Excluded middle is exactly dual. □

Proposition 2: Cut is eliminable in all BS systems.

Proof: I will only sketch the basic proof strategy and show the crucial case. The proof is a double induction with a primary induction is on Cut formula weight and a secondary induction on Cut height. We show first by induction on Cut height that Cut on atomics is eliminable. To do this, we show, first, that Cut on axioms is elinimable, and, second, that, in all applications of Cut where Cut formula is not principal (where, if it's atomic, it never will be), the application of Cut be pushed up the proof tree. This induction then serves as our base case for the primary induction on formula weight. For the inductive step, we show that either Cut height can be reduced, or, in the crucial cases where the Cut formula is principal in both premises, the weight of the Cut formula can be reduced. For the $\boldsymbol{c}$ rules, we have the following reduction:

$$\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \Delta}{\Gamma, \boldsymbol{c}\langle A \circ B\rangle \vdash \Delta}\,\boldsymbol{c}_{\circ_L} \quad \dfrac{\Gamma \vdash \boldsymbol{a}\langle A\rangle, \Delta \quad \Gamma \vdash \boldsymbol{b}\langle B\rangle, \Delta}{\Gamma \vdash \boldsymbol{c}\langle A \circ B\rangle, \Delta}\,\boldsymbol{c}_{\circ_R}}{\Gamma \vdash \Delta}\,\text{Cut}$$

$$\rightsquigarrow$$

$$\dfrac{\dfrac{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \Delta \quad \Gamma \vdash \boldsymbol{a}\langle A\rangle, \Delta}{\Gamma, \boldsymbol{b}\langle B\rangle \vdash \Delta}\,\text{Cut} \quad \Gamma \vdash \boldsymbol{b}\langle B\rangle, \Delta}{\Gamma \vdash \Delta}\,\text{Cut}$$

The reduction for the $\boldsymbol{c}^*$ rules is exactly dual. □

Proposition 3: Weakening is eliminable in all BS systems.

Proof: Weakening can be derived directly from Cut, given Reflexivity.[27] Where Δ has n signed sentences, we derive Weakening on the left as follows:

$$\dfrac{\Gamma \vdash \Delta \quad \overline{\Gamma, \Gamma', \Delta \vdash \Delta}\,\text{Reflex.}}{\Gamma, \Gamma' \vdash \Delta}\,n \text{ applications of Cut}$$

Weakening on the right is derived similarly. Since Weakening is eliminable given Cut, and Cut is eliminable, Weakening is eliminable.

6.3 Soundness and Completeness

Proposition 4: *Soundness of BS_{FDE}:* If BS_{FDE} proves $\Gamma \vdash \Delta$, then $\Gamma \vDash_{\text{FDE}} \Delta$

Proof: Straightforward by induction. For the base case, any instance of the axiom schema $\Gamma, \varphi \vdash \varphi, \Delta$ is valid, since, for any valuation, if φ is correct, then φ is not incorrect. For the inductive step, we show that our rules preserve validity. The case of negation is obvious. For the binary connective schema, suppose we have $\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \Delta$ at height n and derive $\Gamma, \boldsymbol{c}\langle A \circ B\rangle \vdash \Delta$ at height $n+1$. By our inductive hypothesis, there's no valuation such that all the stances in Γ are correct, $[\boldsymbol{a}] \in v(A)$, $[\boldsymbol{b}] \in v(B)$, and all of the stances in Δ are incorrect. Since $[\boldsymbol{c}] \in v(A \circ B)$, just in case $[\boldsymbol{a}] \in v(A)$, and $[\boldsymbol{b}] \in v(B)$ there's no valuation such that all the stances in Γ are correct, $[\boldsymbol{c}] \in v(A \circ B)$, and all of the stances in Δ are incorrect. So, if $\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vDash \Delta$,

[27] A direct proof of height-preserving Weakening elimination can be given (cf. Negri and von Plato [25]). I present this simple proof using Cut just for ease of proof.

then $\Gamma, \boldsymbol{c}\langle A \circ B \rangle \vDash \Delta$. The other cases are similar. □

Proposition 5: *Completeness of BS_{FDE}:* If $\Gamma \vDash_{\mathrm{B_{FDE}}} \Delta$, then $\mathrm{BS_{FDE}}$ proves $\Gamma \vdash \Delta$.

Proof: We prove the contrapositive. Suppose $\Gamma \vdash \Delta$ is not provable. We will consider a *reduction tree* that extends $\Gamma \vdash \Delta$ in a number of steps using the following procedure:[28]

1. If $+\langle \neg A \rangle \in \Gamma$ and $-\langle A \rangle \notin \Gamma$, extend $\Gamma \vdash \Delta$ to $\Gamma, -\langle A \rangle \vdash \Delta$.
2. If $-\langle \neg A \rangle \in \Gamma$ and $+\langle A \rangle \notin \Gamma$, extend $\Gamma \vdash \Delta$ to $\Gamma, +\langle A \rangle \vdash \Delta$
3. If $+\langle \neg A \rangle \in \Delta$ and $-\langle A \rangle \notin \Delta$, extend $\Gamma \vdash \Delta$ to $\Gamma \vdash -\langle A \rangle, \Delta$.
4. If $-\langle \neg A \rangle \in \Delta$ and $+\langle A \rangle \notin \Delta$, extend $\Gamma \vdash \Delta$ to $\Gamma \vdash +\langle A \rangle, \Delta$
5. If $\boldsymbol{c}\langle A \circ B \rangle \in \Gamma$, and either $\boldsymbol{a}\langle A \rangle \notin \Gamma$ or $\boldsymbol{b}\langle B \rangle \notin \Gamma$, extend $\Gamma \vdash \Delta$ to $\Gamma, \boldsymbol{a}\langle A \rangle, \boldsymbol{b}\langle B \rangle \vdash \Delta$
6. If $\boldsymbol{c}^*\langle A \circ B \rangle \in \Gamma$ and neither $\boldsymbol{a}^*\langle A \rangle \in \Gamma$ nor $\boldsymbol{b}^*\langle B \rangle \in \Gamma$, extend $\Gamma \vdash \Delta$ into two branches: $\Gamma, \boldsymbol{a}^*\langle A \rangle \vdash \Delta$ and $\Gamma, \boldsymbol{b}^*\langle B \rangle \vdash \Delta$ and continue the procedure on each.
7. If $\boldsymbol{c}\langle A \circ B \rangle \in \Delta$ and neither $\boldsymbol{a}\langle A \rangle \in \Delta$ nor $\boldsymbol{b}\langle B \rangle \in \Delta$, extend $\Gamma \vdash \Delta$ into two branches: $\Gamma \vdash \boldsymbol{a}\langle A \rangle, \Delta$ and $\Gamma \vdash \boldsymbol{b}\langle B \rangle, \Delta$ and continue the procedure on each.
8. If $\boldsymbol{c}^*\langle A \circ B \rangle \in \Delta$, and either $\boldsymbol{a}^*\langle A \rangle \notin \Delta$ or $\boldsymbol{b}^*\langle B \rangle \notin \Delta$, extend $\Gamma \vdash \Delta$ to $\Gamma \vdash \boldsymbol{a}^*\langle A \rangle, \boldsymbol{b}^*\langle B \rangle, \Delta$

Since $\Gamma \vdash$ contains a finite number of formulas of finite length, there is a finite number of steps we must take until this procedure terminates.

Now, suppose we have constructed a reduction tree for $\Gamma \vdash \Delta$. The are two possibilities. The first possibility is that each of the final sequents in the reduction tree is of the form $\Gamma', \varphi \vdash \varphi, \Delta'$ for some atomic formula φ. In this case, $\Gamma \vdash \Delta$ is provable. To see this, note that each of these final sequents is an instance of the axiom schema, and now see that we can now run the extension procedure we just did in reverse to arrive at a proof of $\Gamma \vdash \Delta$. Contradiction, so there must be some final sequent $\Gamma' \vdash \Delta'$ in reduction tree of $\Gamma \vdash \Delta$ that such that there is no atomic formula φ such that $\varphi \in \Gamma$ and $\varphi \in \Delta$.

We now show by induction on formula complexity that there is an FDE valuation v such that

[28] This method of reduction trees is deployed by Ripley [33] for ST, based on Takeuti [45]. A very similar method involves the construction of saturated sets, spelled out in detail in [20, 32-35]. See [44] for a proof of this sort for signed tableaux using a similar schematization strategy to the one deployed here.

if $+\langle A\rangle \in \Gamma'$, then $1 \in v(A)$

if $-\langle A\rangle \in \Gamma'$, then $0 \in v(A)$

if $+\langle A\rangle \in \Delta'$, then $1 \notin v(A)$

if $-\langle A\rangle \in \Delta'$, then $0 \notin v(A)$

For the base case, we define an FDE valuation v such that, for all atomics p, $1 \in v(p)$ just in case $+\langle p\rangle \in \Gamma'$, $0 \in v(p)$ just in case $-\langle A\rangle \in \Gamma'$, $+\langle p\rangle \in \Delta'$ just in case $1 \notin v(p)$, and $-\langle p\rangle \in \Delta'$ just in case $0 \notin v(p)$. Since there is no atomic signed formula φ such that $\varphi \in \Gamma$ and $\varphi \in \Delta$, there exists such a valuation. For the inductive step, we suppose the above condition holds for formulas of complexity n and show that it holds for formulas of complexity $n + 1$.

Consider first the case in which A is of the form $\neg B$. If $+\langle\neg B\rangle \in \Gamma'$, then, by rule 1 of the reduction procedure, $-\langle B\rangle \in \Gamma'$. By the inductive hypothesis, $0 \in v(B)$, and so $1 \in v(\neg B)$. The same reasoning goes if $-\langle\neg B\rangle \in \Gamma'$. Suppose now that $+\langle\neg B\rangle \in \Delta'$. By Rule 3 of the reduction procedure $-\langle B\rangle \in \Delta'$. By the inductive hypothesis $0 \notin v(B)$, and so $1 \notin v(\neg B)$. The same reasoning goes if $-\langle\neg B\rangle \in \Delta'$.

Now suppose that A is of the form $B \circ C$, for any binary connective $\circ$. Here, to establish the above condition for any connective, we will show, for any stance $\boldsymbol{c}$:

if $\boldsymbol{c}\langle A\rangle \in \Gamma'$, then $[\boldsymbol{c}] \in v(A)$

if $\boldsymbol{c}^*\langle A\rangle \in \Gamma'$, then $[\boldsymbol{c}^*] \in v(A)$

if $\boldsymbol{c}\langle A\rangle \in \Delta'$, then $[\boldsymbol{c}] \notin v(A)$

if $\boldsymbol{c}^*\langle A\rangle \in \Delta'$, then $[\boldsymbol{c}^*] \notin v(A)$

Given that $\boldsymbol{c}$ is either + or −, and whichever one it is, $\boldsymbol{c}^*$ is the other, establishing this suffices to establish the above condition. So, there are four cases to consider. Suppose first $\boldsymbol{c}\langle B \circ C\rangle \in \Gamma'$. Then, by rule 5 of the reduction procedure, $\boldsymbol{a}\langle B\rangle \in \Gamma'$ and $\boldsymbol{b}\langle C\rangle \in \Gamma'$. By the inductive hypothesis $[\boldsymbol{a}] \in v(B)$ and $[\boldsymbol{b}] \in v(C)$, and so $[\boldsymbol{c}] \in v(B \circ C)$. Suppose now $\boldsymbol{c}\langle B \circ C\rangle \in \Delta'$. By rule 7 of the reduction procedure, either $\boldsymbol{a}\langle B\rangle \in \Delta'$ or $\boldsymbol{b}\langle C\rangle \in \Delta'$. By the inductive hypothesis, if $\boldsymbol{a}\langle B\rangle \in \Delta'$, then $[\boldsymbol{a}^*] \notin v(B)$ and, if $\boldsymbol{b}\langle C\rangle \in \Delta'$, then $[\boldsymbol{b}^*] \notin v(C)$, and so, by the valuation function, $[\boldsymbol{c}] \notin v(B \circ C)$. The cases where $\boldsymbol{c}^*\langle B \circ C\rangle \in \Gamma'$ or $\boldsymbol{c}^*\langle B \circ C\rangle \in \Delta'$ are similar.

So, there is a valuation v such that $1 \in v(A)$ for all formulas of the form $+\langle A\rangle \in \Gamma'$ and $0 \in v(A)$ for all formulas of the form $-\langle A\rangle \in \Gamma'$, $1 \notin v(A)$ for all formulas of the form $+\langle A\rangle \in \Delta'$ and $0 \notin v(A)$ for all formulas of the form $-\langle A\rangle \in \Delta'$. That is, there is a valuation v such that all of the stances in Γ' are correct and all of the stances in Δ' are incorrect. That is, $\Gamma' \nvDash \Delta'$. But $\Gamma' \supseteq \Gamma$ and $\Delta' \supseteq \Delta$, so this is true of Γ and Δ as well. Thus, $\Gamma \nvDash \Delta$. □.

Proposition 6: *Soundness and Completeness for BS_{K3}*: $\Gamma \vDash_{B_{K3}} \Delta$ just in case BS_{K3} proves $\Gamma \vdash \Delta$.

Soundness: Same as soundness for BS_{FDE} except we also note that any instance $\Gamma', \varphi, \varphi^* \vdash \Delta$ is valid, since there is can be no K3 valuation where φ and φ^* are correct, since K3 valuations cannot contain $\{1, 0\}$. □

Completeness: Same as completeness BS_{FDE} except that we also consider the case in which each of the final sequents in the reduction tree is of the form $\Gamma', \varphi \vdash \varphi, \Delta'$ or $\Gamma', \varphi, \varphi^* \vdash \Delta$ for some atomic formula φ. Once again, if this is so, then $\Gamma \vdash \Delta$ would be provable, and so there must be some final sequent $\Gamma' \vdash \Delta'$ in reduction tree of $\Gamma \vdash \Delta$ that such that, for all atomic formulas φ, neither $\varphi \in \Gamma'$ and $\varphi \in \Delta'$ nor $\varphi, \varphi^* \in \Gamma'$. Then, for the base case of the induction, we know that there is a K3 valuation v such that, for all atomics p, $1 \in v(p)$ just in case $+\langle p \rangle \in \Gamma'$ and $0 \in v(p)$ just in case $-\langle p \rangle \in \Gamma'$, since we know that it can't be the case that $+\langle p \rangle \in \Gamma'$ and $-\langle p \rangle \in \Gamma'$. Everything else proceeds the same. □

Proposition 7: *Soundness and Completeness for BS_{LP}*: $\Gamma \vDash_{B_{LP}} \Delta$ just in case BS_{LP} proves $\Gamma \vdash \Delta$.

Tweaks to the soundness and completeness proof for BS_{FDE} are exactly analogous to those made for BS_{K3} □

Proposition 8: All BN systems are sound and complete.

Soundness: Follows from the derivability of all of the rules in the BS systems. We already showed axiom schemas are derivable. The introduction rules of the ND system are directly derivable from the right rules of this sequent system (through a single application of Weakening in the case of the $\boldsymbol{c}^*{}_\circ$ rules). The derivability of the elimination rules follows from the invertibility of the rules:

$$\dfrac{\dfrac{\dfrac{}{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \boldsymbol{a}\langle A\rangle}\text{Reflex.}}{\Gamma, \boldsymbol{c}\langle A \circ B\rangle \vdash \boldsymbol{a}\langle A\rangle}\boldsymbol{c}^*{}_\circ \qquad \Gamma \vdash \boldsymbol{c}\langle A \circ B\rangle}{\Gamma \vdash \boldsymbol{a}\langle A\rangle}\text{Cut} \qquad \dfrac{\dfrac{\dfrac{}{\Gamma, \boldsymbol{a}\langle A\rangle, \boldsymbol{b}\langle B\rangle \vdash \boldsymbol{b}\langle B\rangle}\text{Reflex.}}{\Gamma, \boldsymbol{c}\langle A \circ B\rangle \vdash \boldsymbol{b}\langle B\rangle}\boldsymbol{c}^*{}_\circ \qquad \Gamma \vdash \boldsymbol{c}\langle A \circ B\rangle}{\Gamma \vdash \boldsymbol{b}\langle B\rangle}\text{Cut}$$

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash \boldsymbol{c}^*\langle A \circ B\rangle \qquad \dfrac{\dfrac{}{\Gamma, \boldsymbol{a}^*\langle A\rangle \vdash \boldsymbol{a}^*\langle A\rangle, \boldsymbol{b}^*\langle B\rangle}\,\text{Reflex.} \qquad \dfrac{}{\Gamma, \boldsymbol{b}^*\langle B\rangle \vdash \boldsymbol{a}^*\langle A\rangle, \boldsymbol{b}^*\langle B\rangle}\,\text{Reflex.}}{\Gamma, \boldsymbol{c}^*\langle A \circ B\rangle \vdash \boldsymbol{a}^*\langle A\rangle, \boldsymbol{b}^*\langle B\rangle}\,\boldsymbol{c}^*\circ}{\Gamma \vdash \boldsymbol{a}^*\langle A\rangle, \boldsymbol{b}^*\langle B\rangle}\,\text{Cut} \qquad \Gamma, \boldsymbol{a}^*\langle A\rangle \vdash \varphi}{\Gamma \vdash \boldsymbol{b}^*\langle B\rangle, \varphi}\,\text{Cut} \qquad \Gamma, \boldsymbol{b}^*\langle B\rangle \vdash \varphi}{\Gamma \vdash \varphi}\,\text{Cut}$$

Completeness: Priest [29] has provided unilateral natural deduction systems for FDE, LP, and K3, that are sound and complete with respect to their respective semantics.[29] It is straightforward (albeit a bit tedious) to adapt and generalize these proofs for the bilateral natural deduction systems here, just as I have done so for the BS systems above. For reasons of space, I omit the full presentation of the proof here. □.

6.4 Relations Between Bilateral and Unilateral Consequence

Proposition 9: $+\langle X\rangle \vDash_{B_V} +\langle Y\rangle$ just in case $X \vDash_V Y$.

Proof: Immediate from the definitions. □

Proposition 10: $+\langle X\rangle, -\langle Y\rangle \vDash_{B_{\text{K3}}}$ just in case $X \vDash_{\text{CL}} Y$.

Proof: For the left to right direction, suppose $+\langle X\rangle, -\langle Y\rangle \vDash_{B_{\text{K3}}}$ but $X \nvDash_{\text{CL}} Y$. Then there is no $v \in$ K3 such that $1 \in v(A)$ for all $A \in X$ and $0 \in v(B)$ for all $B \in Y$ but there is some $v \in$ CL such that $1 \in v(A)$ for all $A \in X$ and $1 \notin v(B)$ for all $B \in Y$. This CL valuation will be such that $v(A) = \{1\}$ for all $A \in X$ and $v(B) = \{0\}$ for all $B \in Y$. But that's a K3 valuation in which the above condition holds. Contradiction. So, if $+\langle X\rangle, -\langle Y\rangle \vDash_{B_{\text{K3}}}$, then $X \vDash_{\text{CL}} Y$. For the right to left direction, suppose $X \vDash_{\text{CL}} Y$, but $+\langle X\rangle, -\langle Y\rangle \nvDash_{B_{\text{K3}}}$. Then there is no $v \in$ CL such that $1 \in v(A)$ for all $A \in X$ and $1 \notin v(B)$ for all $B \in Y$ but there is some $v \in$ K3 such that $1 \in v(A)$ for all $A \in X$ and $0 \in v(B)$ for all $B \in Y$. This K3 valuation will be such that $v(A) = \{1\}$ for all $A \in X$ and $v(B) = \{0\}$ for all $B \in Y$. Now consider the classicalization of this valuation v' that aligns with v on atomics except that $v'(p) = \{0\}$ if $v(p) = \varnothing$. It is straightforward to show by induction on the complexity the formulas in X and Y that v' is a CL valuation in which the above condition holds. Contradiction. So, $X \vDash_{\text{CL}} Y$, then $+\langle X\rangle, -\langle Y\rangle \vDash_{B_{\text{K3}}}$. □

Proposition 11: $\vDash_{B_{\text{LP}}} -\langle X\rangle, +\langle Y\rangle$ just in case $X \vDash_{\text{CL}} Y$.

Proof: Proceeds exactly analogously to the proof above. □

[29]See also [46] and [12, 79-82]

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